6 | VECTOR CALCULUS

Chapter Outline

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6.1 | Vector Fields

vector functions

Examples of Vector Fields

Figure 6.2(a) shows a gravitational field exerted by two astronomical objects, such as a star and a planet or a planet and a moon. At any point in the figure, the vector associated with a point gives the net gravitational force exerted by the two objects on an object of unit mass. The vectors of largest magnitude in the figure are the vectors closest to the larger object. The larger object has greater mass, so it exerts a gravitational force of greater magnitude than the smaller object.

3D

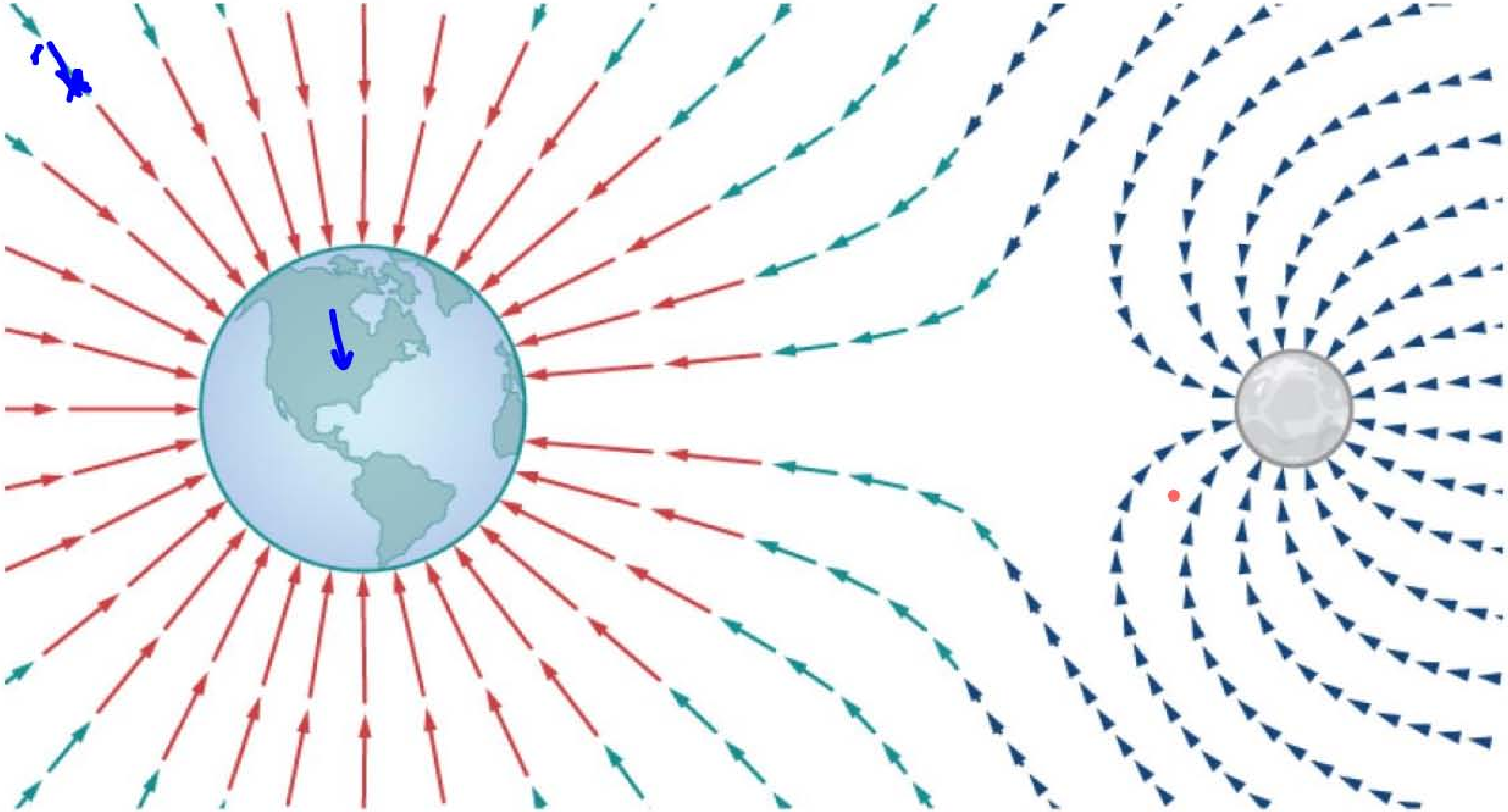
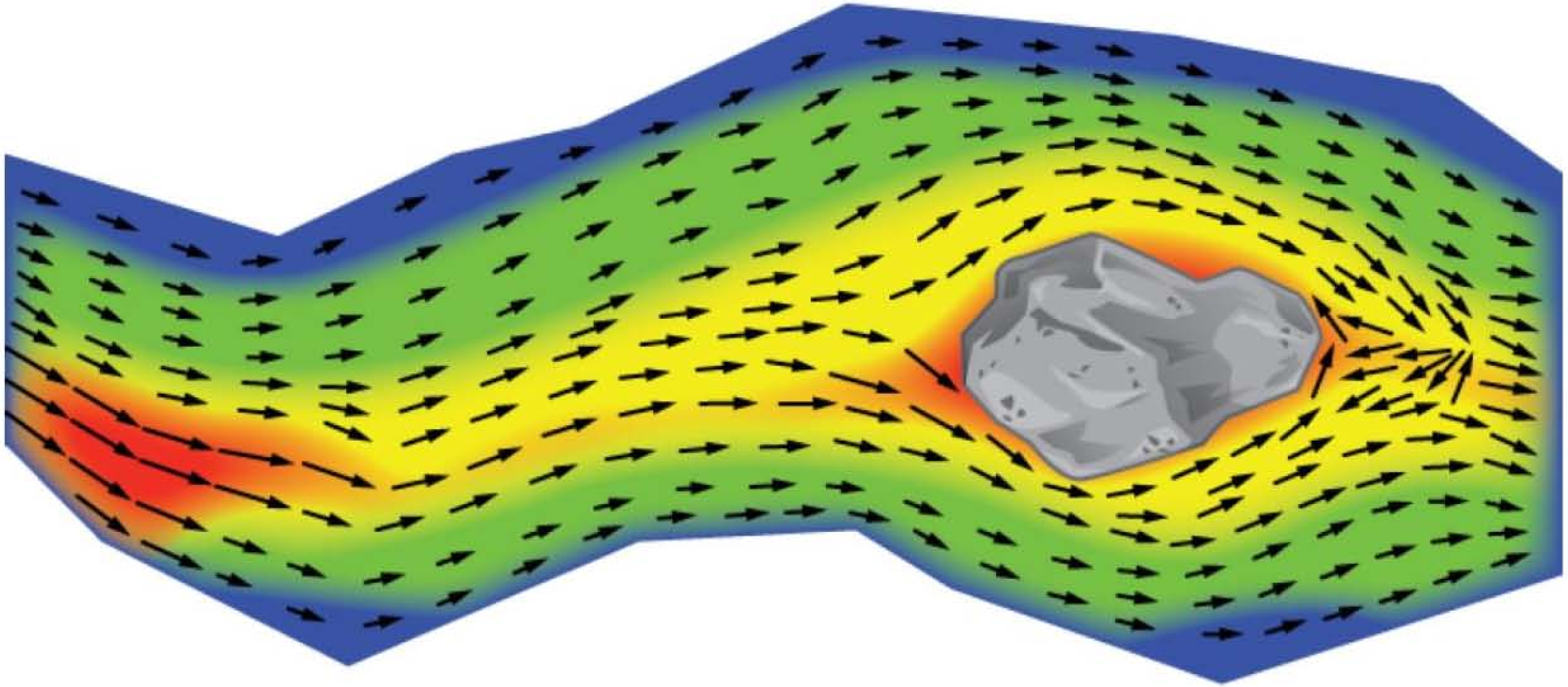


Figure 6.2(b) shows the velocity of a river at points on its surface. The vector associated with a given point on the river's surface gives the velocity of the water at that point. Since the vectors to the left of the figure are small in magnitude, the water is flowing slowly on that part of the surface. As the water moves from left to right, it encounters some rapids around a rock. The speed of the water increases, and a whirlpool occurs in part of the rapids.



Definition

2D vector function

A **vector field F** in $\mathbb{R}^2$ is an assignment of a two-dimensional vector $\mathbf{F}(x, y)$ to each point (x, y) of a subset D of $\mathbb{R}^2$. The subset D is the domain of the vector field.

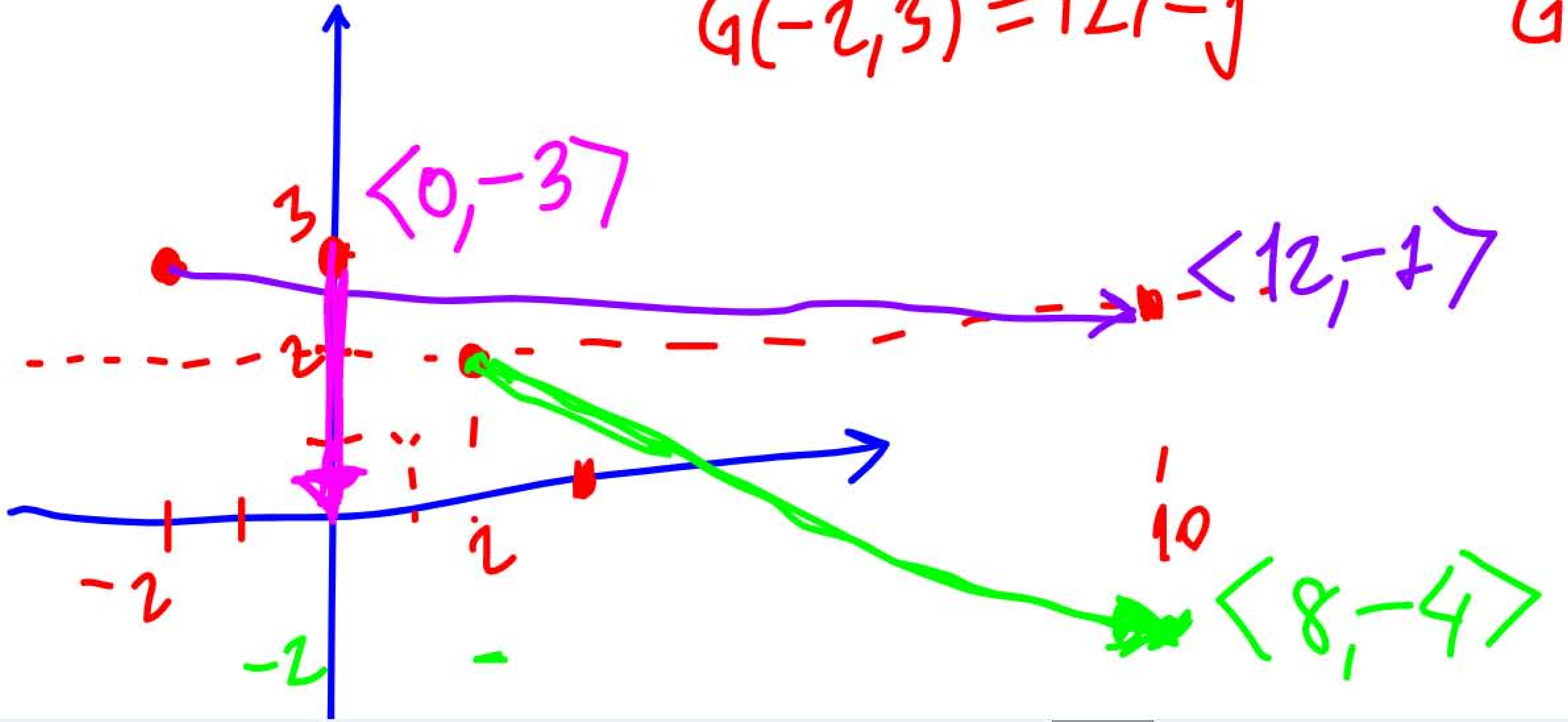
A vector field $\mathbf{F}$ in $\mathbb{R}^3$ is an assignment of a three-dimensional vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) of a subset D of $\mathbb{R}^3$. The subset D is the domain of the vector field. 3D

6.1 Let $\mathbf{G}(x, y) = x^2 y\mathbf{i} - (x + y)\mathbf{j}$ be a vector field in $\mathbb{R}^2$. What vector is associated with the point $(-2, 3)$?

$G(-2,3) = 12i - j$

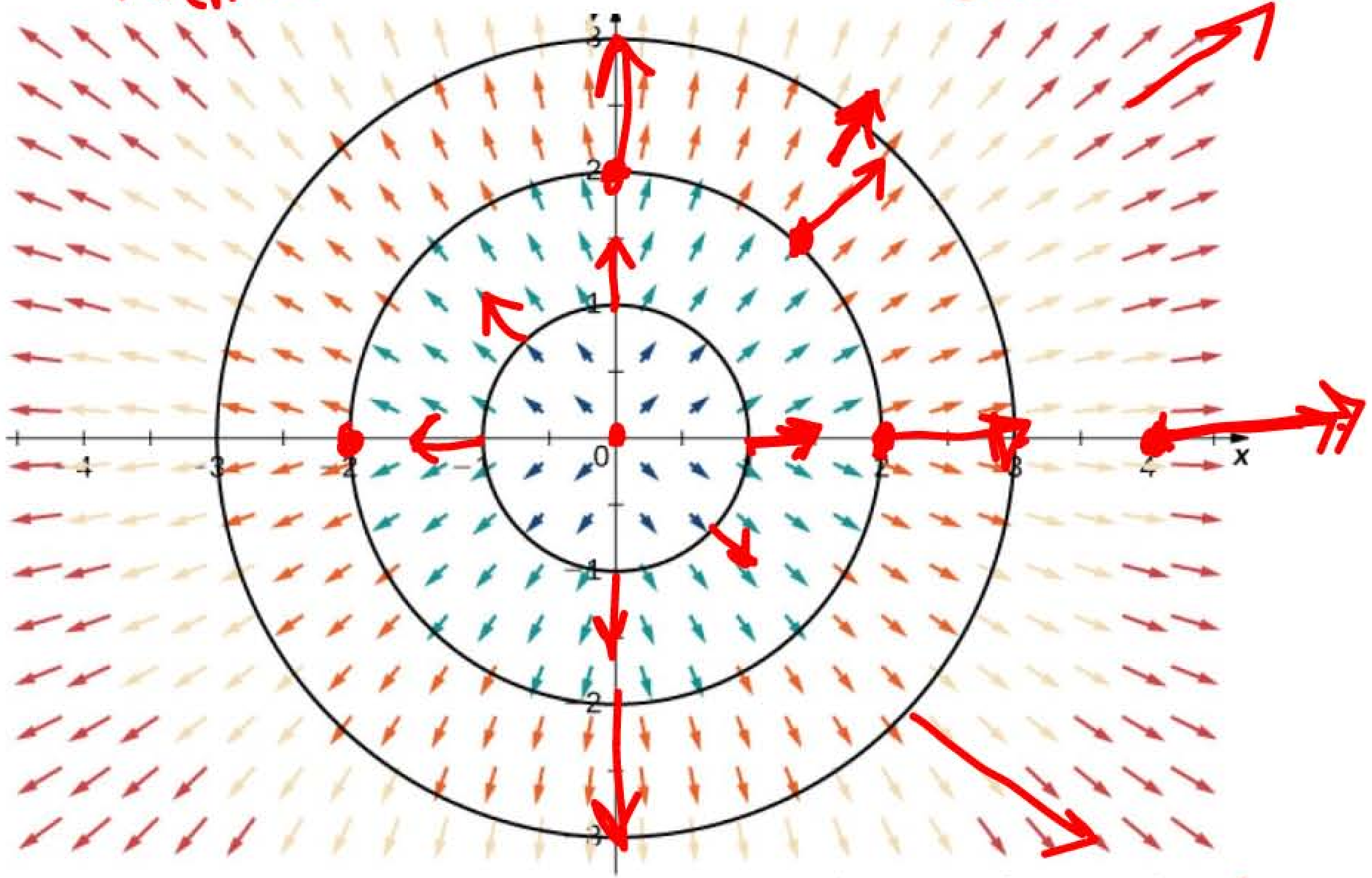
$G(2,2) = 8i - 4j$

$G(0,3) = 0i - 3j$



Sketch the vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$.

radial out



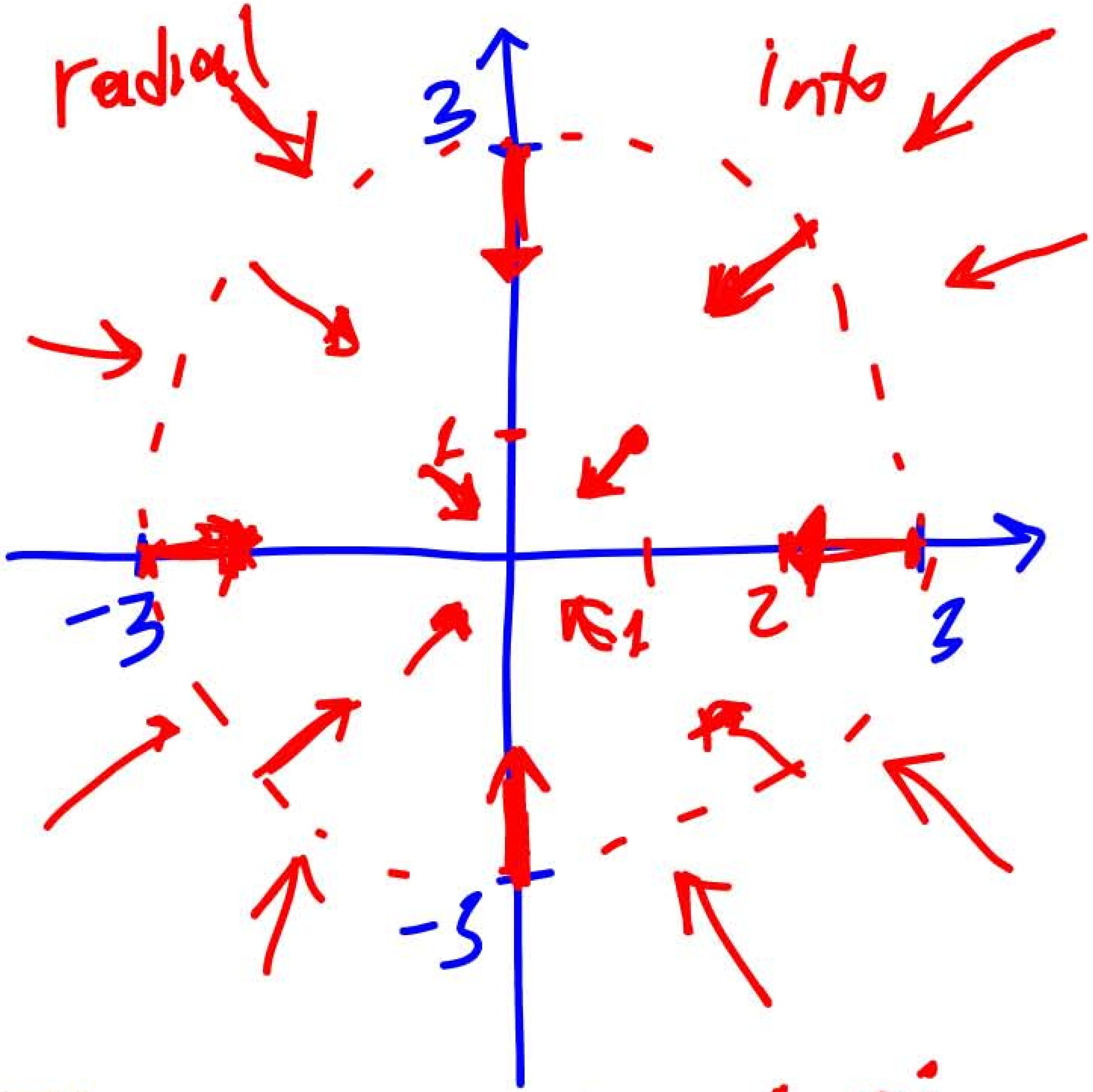
$F(2,2) = \langle 1,1 \rangle$

$F(4,0) = \langle 2,0 \rangle$

$F(2,0) = \langle 1,0 \rangle$

6.2 Draw the radial field $\mathbf{F}(x, y) = -\frac{x}{3}\mathbf{i} - \frac{y}{3}\mathbf{j}$.

radial into



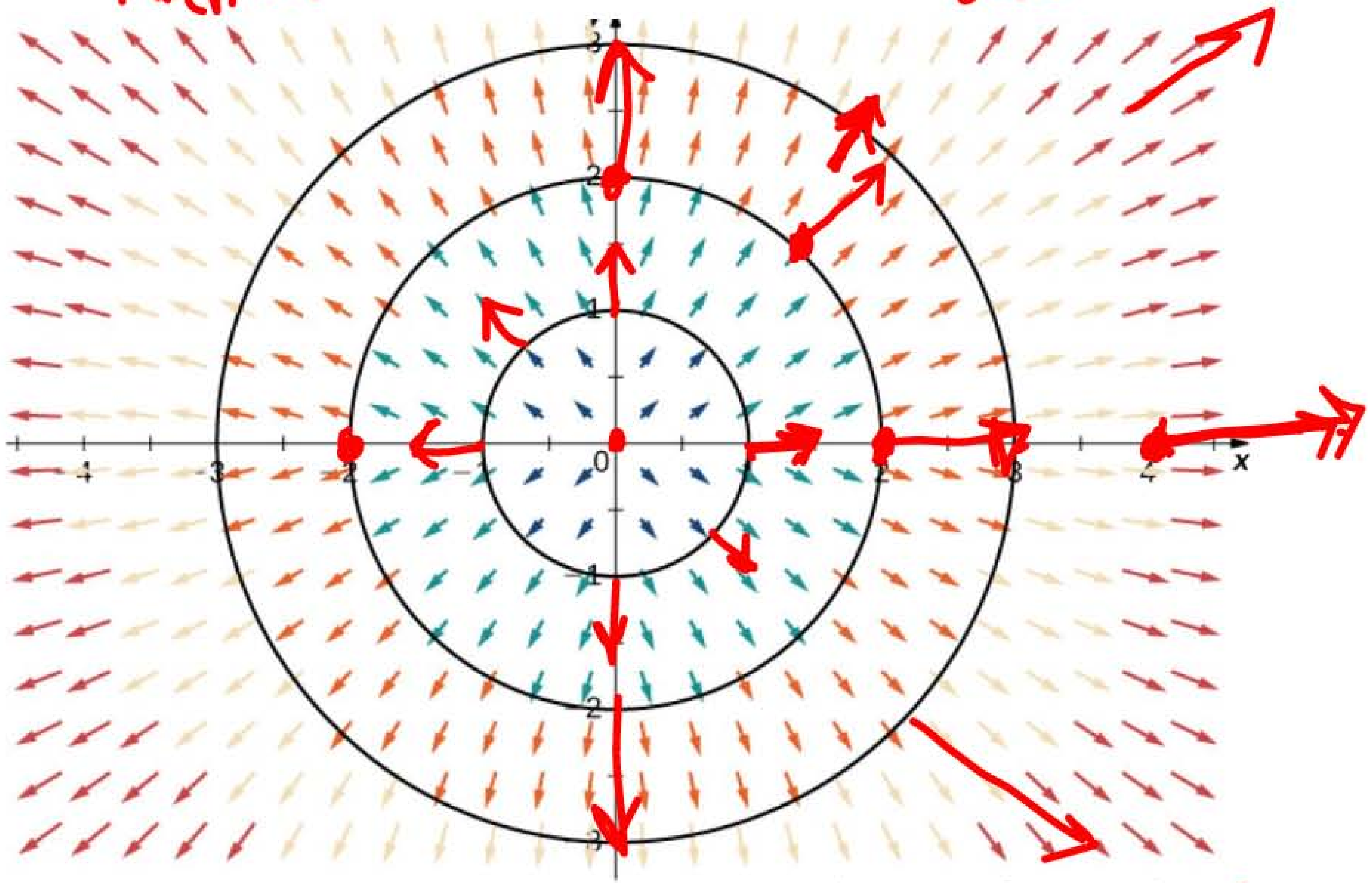
$F(3,0) = \langle -1,0 \rangle = -i + 0j$

$F(0,-3) = \langle 0,1 \rangle = j$

Sketch the vector field $\mathbf{F}(x, y) = \frac{x}{2}\mathbf{i} + \frac{y}{2}\mathbf{j}$.

radial

out



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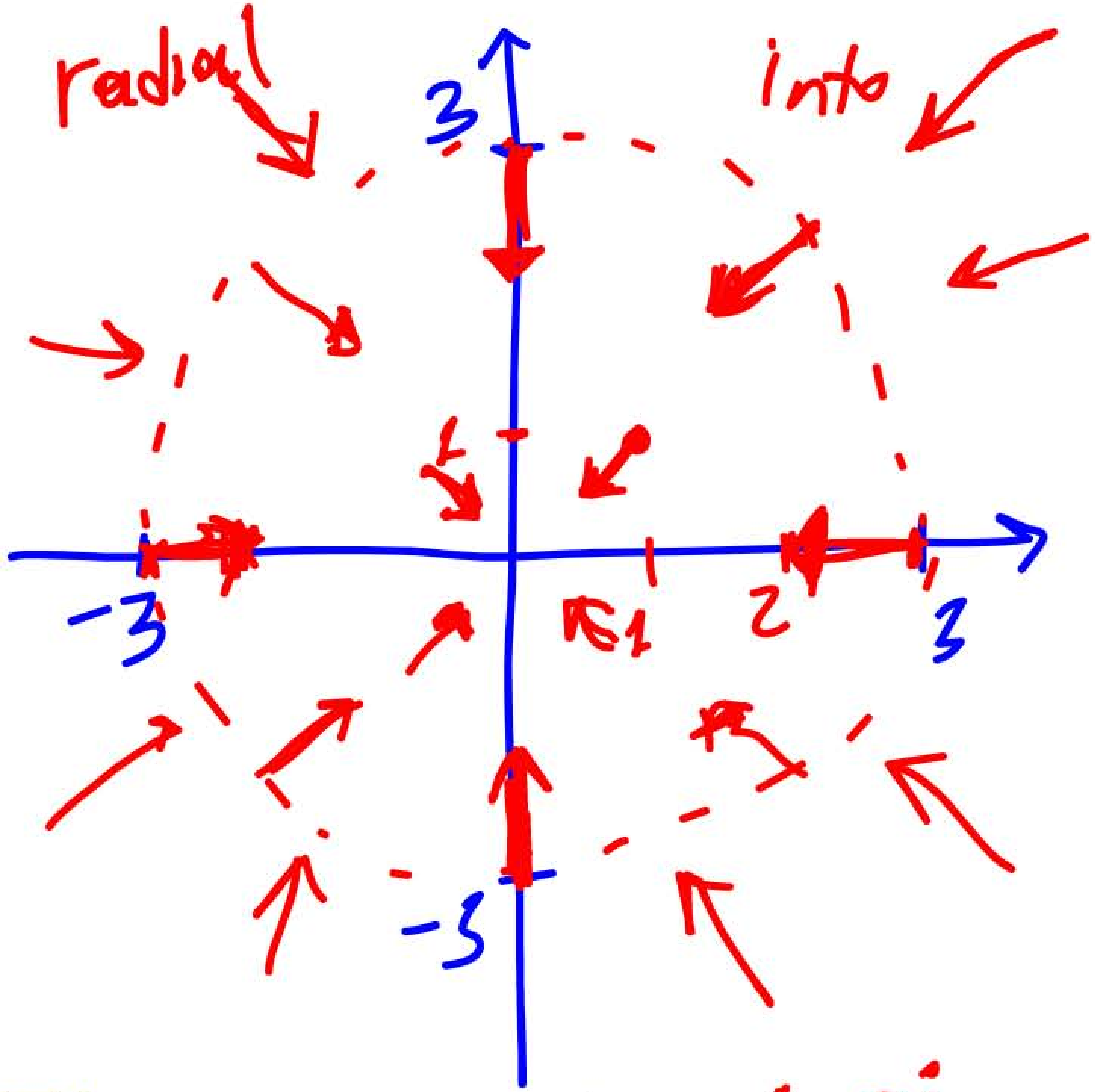
$F(2,0) = \langle 1,0 \rangle$

6.2 Draw the radial field $\mathbf{F}(x, y) = -\frac{x}{3}\mathbf{i} - \frac{y}{3}\mathbf{j}$.

radial

into



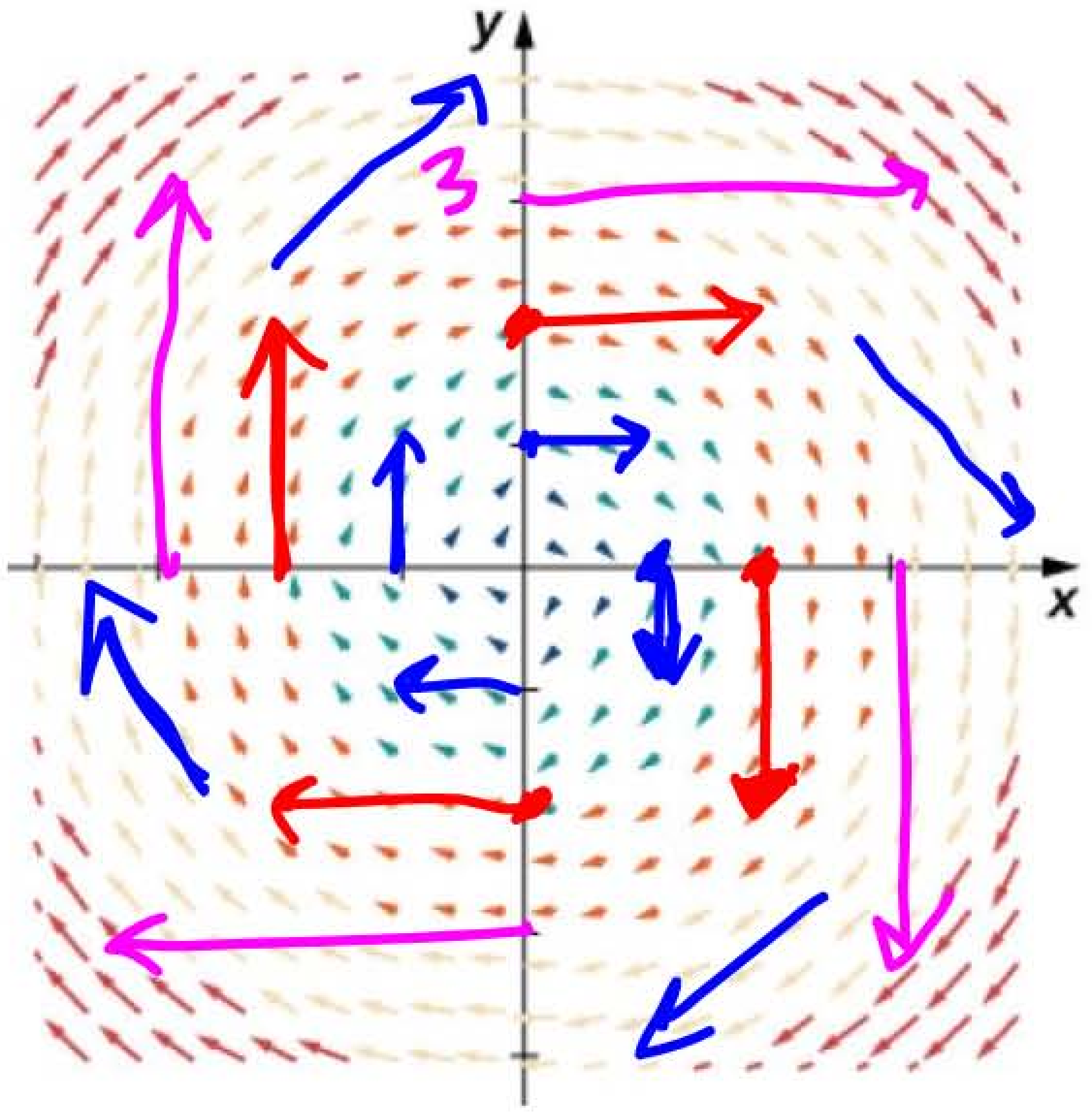
$F(3,0) = \langle -1,0 \rangle = -i + 0j$

$F(0,-3) = \langle 0,1 \rangle = j$

Sketch the vector field $\mathbf{F}(x, y) = \langle y, -x \rangle$.

clockwise rotation

(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$	(x, y)	$\mathbf{F}(x, y)$
$(1, 0)$	$\langle 0, -1 \rangle$	$(2, 0)$	$\langle 0, -2 \rangle$	$(1, 1)$	$\langle 1, -1 \rangle$
$(0, 1)$	$\langle 1, 0 \rangle$	$(0, 2)$	$\langle 2, 0 \rangle$	$(-1, 1)$	$\langle 1, 1 \rangle$
$(-1, 0)$	$\langle 0, 1 \rangle$	$(-2, 0)$	$\langle 0, 2 \rangle$	$(-1, -1)$	$\langle -1, 1 \rangle$
$(0, -1)$	$\langle -1, 0 \rangle$	$(0, -2)$	$\langle -2, 0 \rangle$	$(1, -1)$	$\langle -1, -1 \rangle$



$F(2,0) = \langle 0, -2 \rangle = -2j$

$F(0,2) = \langle 2, 0 \rangle = 2i$

6.3 Sketch vector field $\mathbf{F}(x, y) = \langle -2y, 2x \rangle$. Is the vector field radial, rotational, or neither?

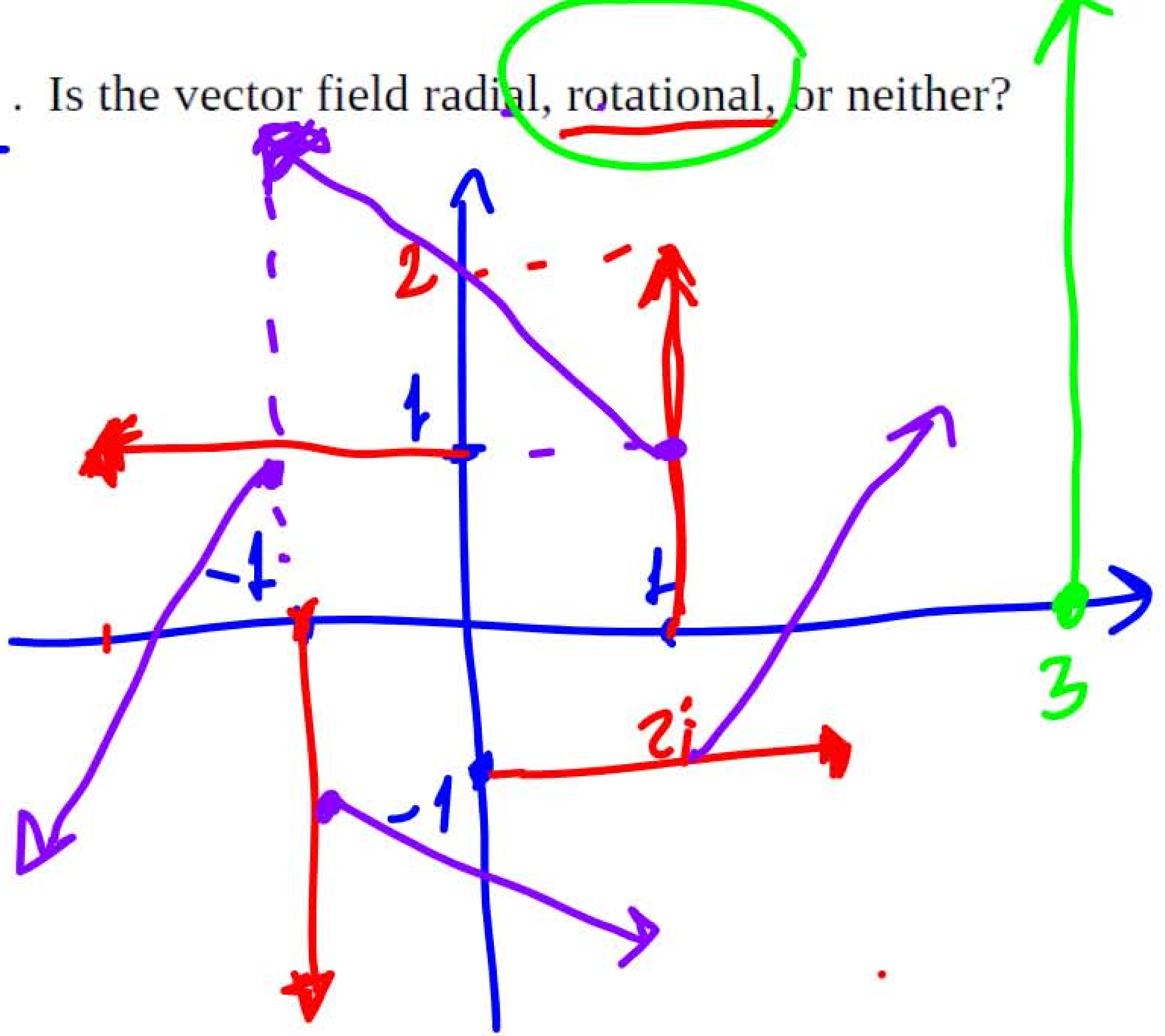
$F(1,0) = \langle 0, 2 \rangle = 2j$

$F(0,1) = \langle -2, 0 \rangle = -2i$

$F(-1,0) = \langle 0, -2 \rangle = -2j$

$F(0,-1) = \langle 2, 0 \rangle = 2i$

$F(1,1) = \langle -2, 2 \rangle$

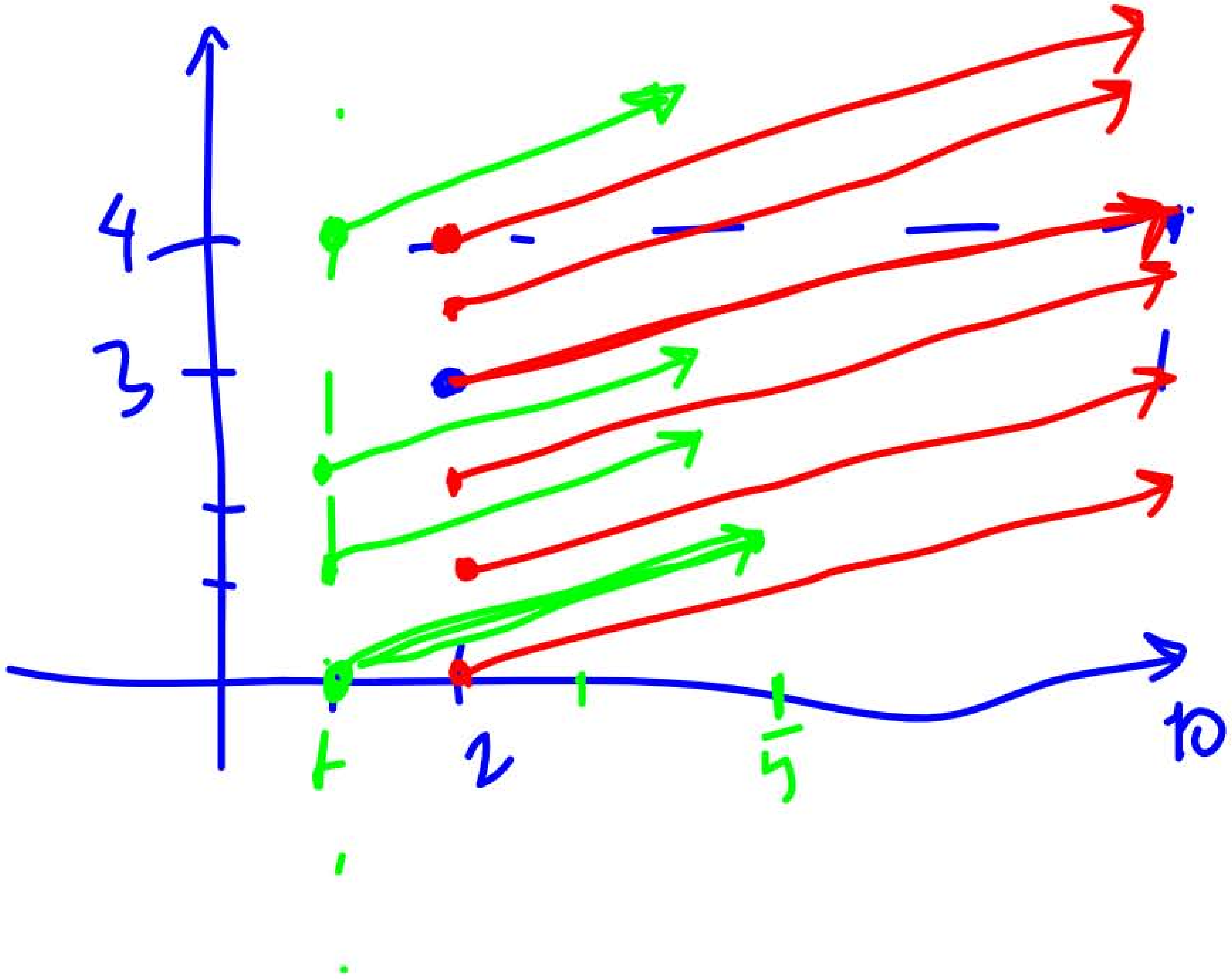


counter clockwise

rotational.

6.4 Vector field $v(x, y) = \langle 4|x|, 1 \rangle$ models the velocity of water on the surface of a river. What is the speed of the water at point $(2, 3)$? Use meters per second as the units.

$$v(2,3) = \langle 8, 1 \rangle \qquad |\langle 8, 1 \rangle| = \sqrt{8^2 + 1^2} = \sqrt{65} \text{ m/s.}$$



Sketching a Vector Field in Three Dimensions

Describe vector field $\mathbf{F}(x, y, z) = \langle 1, 1, z \rangle$.

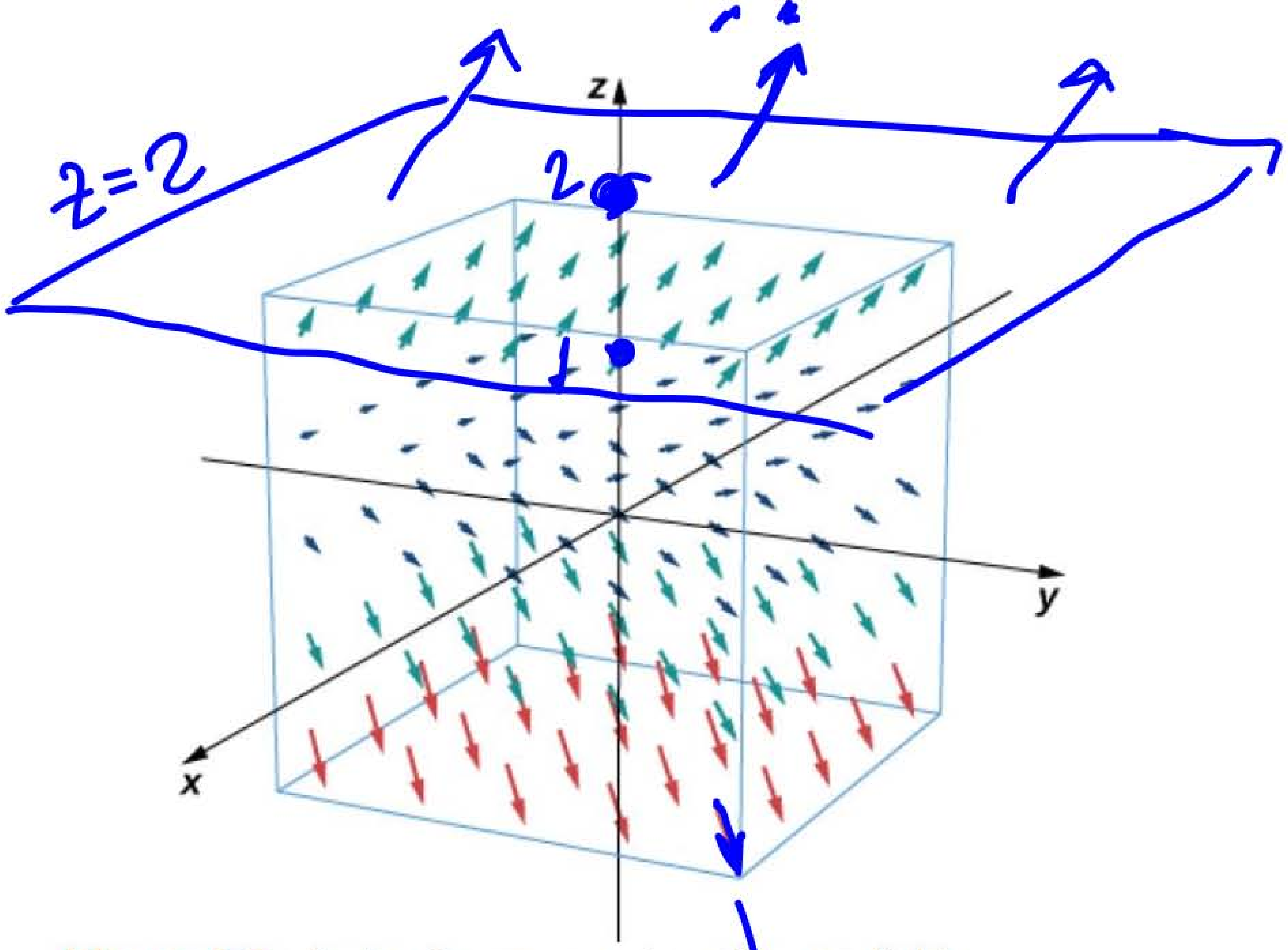


Figure 6.7 A visual representation of vector field $\mathbf{F}(x, y, z) = \langle 1, 1, z \rangle$.

6.6 Sketch vector field $\mathbf{G}(x, y, z) = \langle 2, \frac{z}{2}, 1 \rangle$.

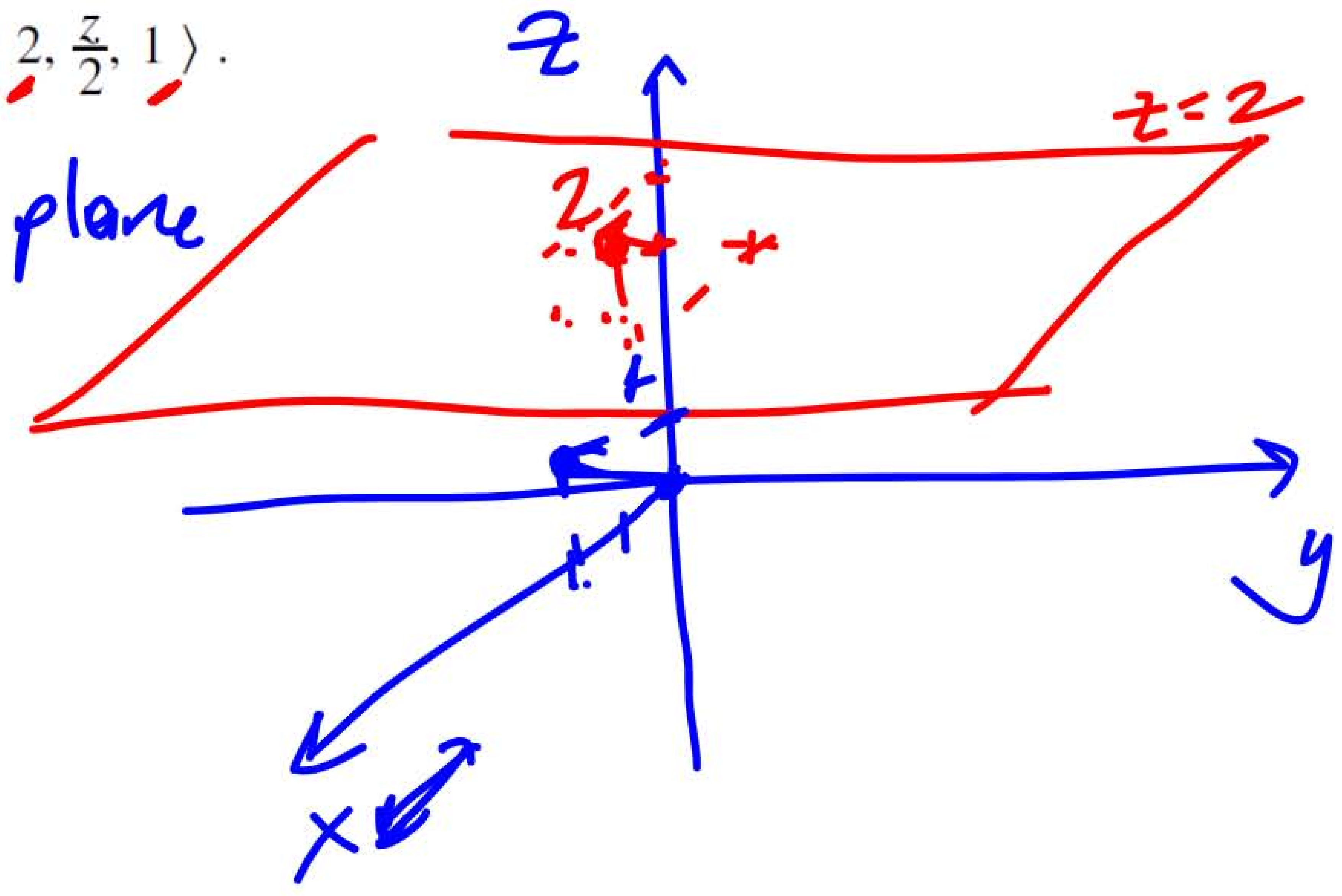
$z = 2$ plane parallel to xy plane

$\langle 2, 1, 1 \rangle$

$G(0,0,0) = \langle 2, 0, 1 \rangle$

all the points in xy plane $(z=0)$

we will have the same vector $\langle 2, 0, 1 \rangle$



Describing a Gravitational Vector Field

Newton's law of gravitation states that $\mathbf{F} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}$

inversely proportional to r

distance to the origin.

the magnitude is same for the points on a sphere

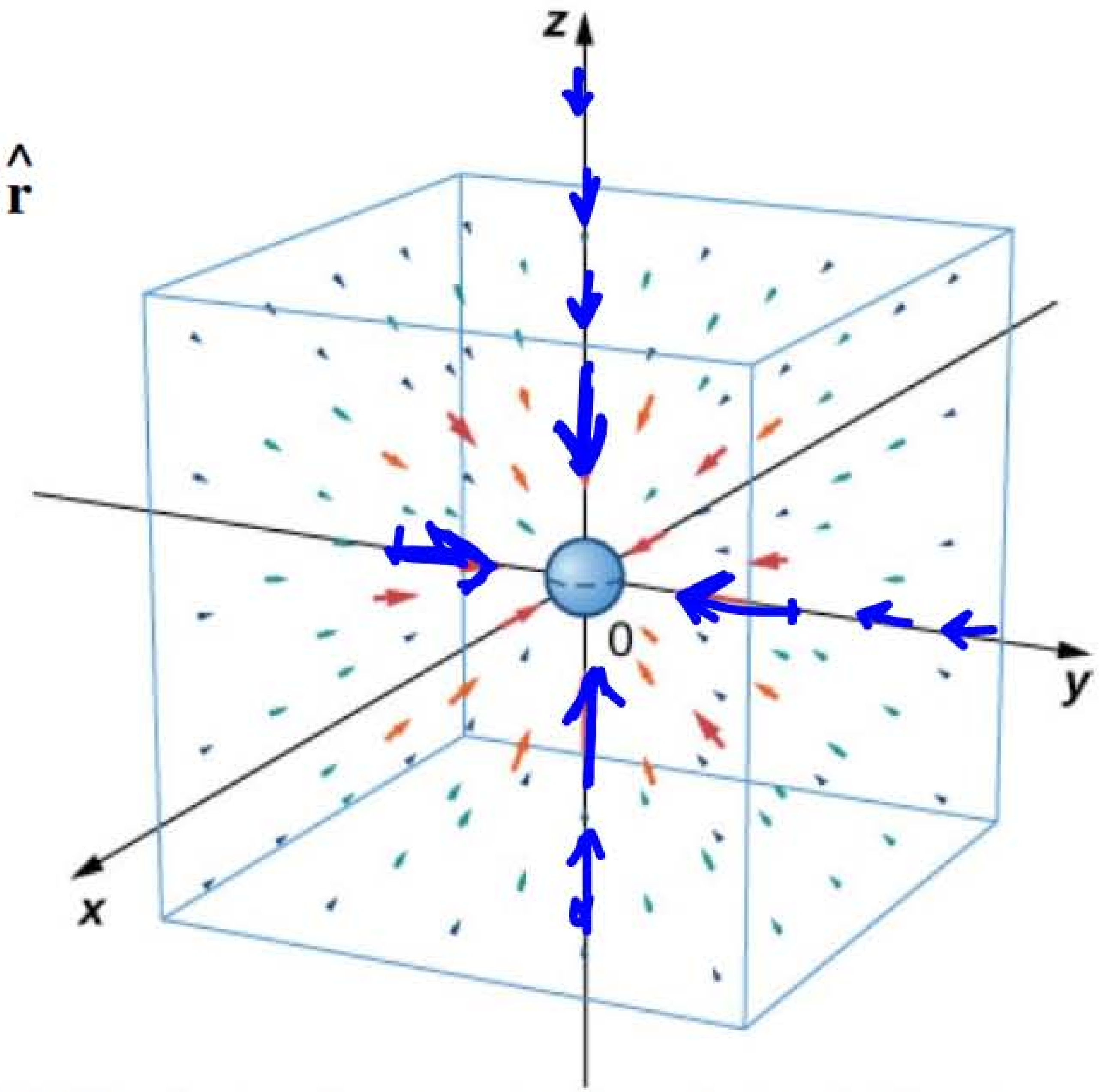


Figure 6.8 A visual representation of gravitational vector field $\mathbf{F} = -Gm_1 m_2 \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle$ for a large mass at the origin.

Definition

$\nabla f = \langle f_x, f_y \rangle$ $\qquad$ $\nabla f = \langle f_x, f_y, f_z \rangle$

A vector field $\mathbf{F}$ in $\mathbb{R}^2$ or in $\mathbb{R}^3$ is a **gradient field** if there exists a scalar function f such that $\nabla f = \mathbf{F}$.

Sketching a Gradient Vector Field

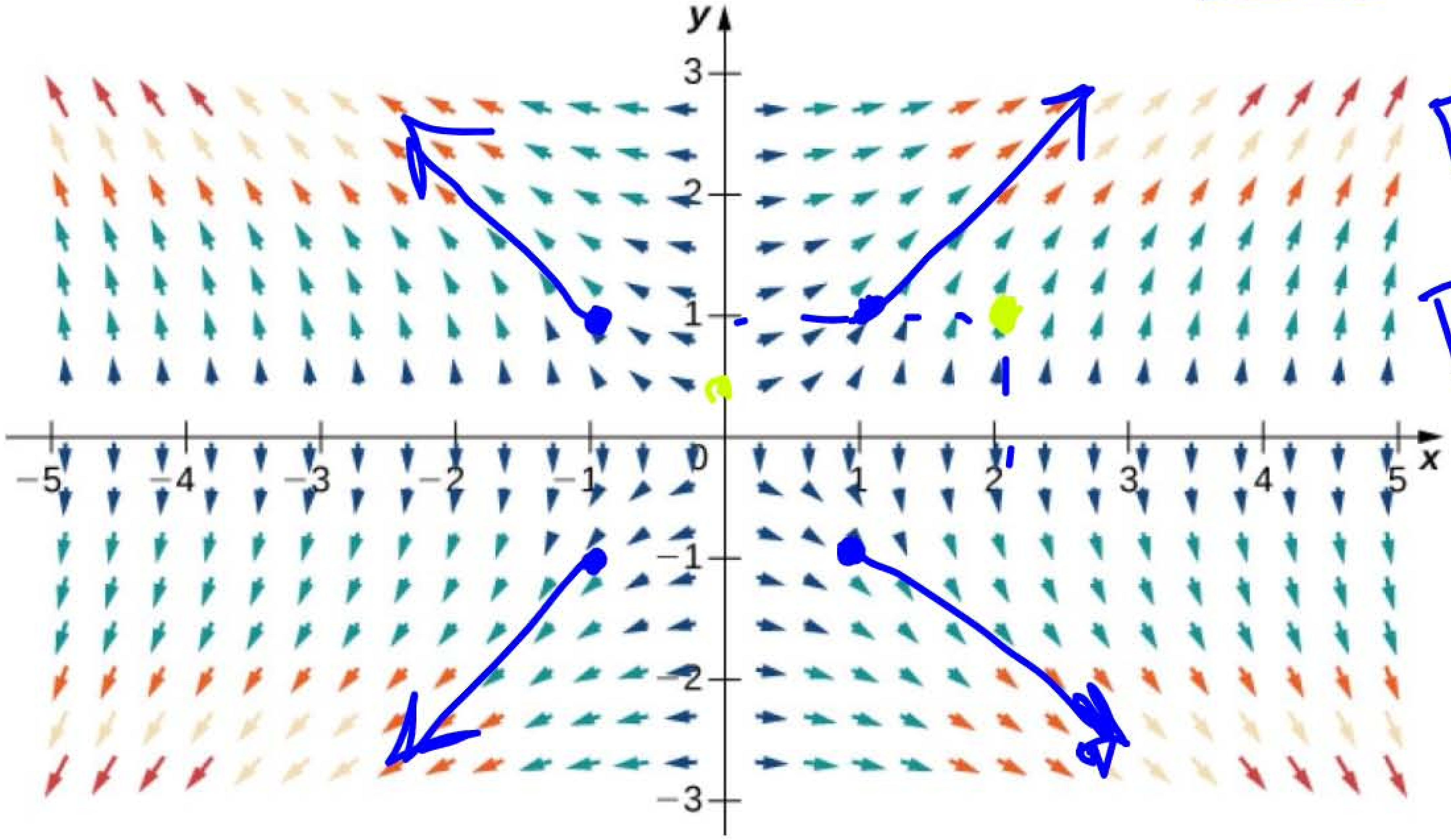
Use technology to plot the gradient vector field of $\cancel{f(x, y) = x^2 y^2}$ $\boxed{f(x, y) = x^2 y^2.}$

$\nabla f = \langle 2xy^2, 2x^2y \rangle$

$\nabla f(1,1) = \langle 2, 2 \rangle$

$\nabla f(-1,0) = \langle 0, 0 \rangle$

$\nabla f(-1,-1) = \langle -2, -2 \rangle$



$\nabla f = \mathbf{F}$. In this situation, f is called a **potential function** for $\mathbf{F}$.

6.10 Verify that $f(x, y) = x^3 y^2 + x$ is a potential function for velocity field $\mathbf{v}(x, y) = \langle 3x^2 y^2 + 1, 2x^3 y \rangle$.

$$\nabla f = \langle f_x, f_y \rangle$$

$$= \langle 3x^2y^2 + 1, 2x^3y \rangle = v(x,y)$$

f is a potential function for v.

Theorem 6.2: The Cross-Partial Property of Conservative Vector Fields

Let **F** be a vector field in two or three dimensions such that the component functions of **F** have continuous second-order mixed-partial derivatives on the domain of **F**.

If $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$ is a conservative vector field in $\mathbb{R}^2$, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. If $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ is a conservative vector field in $\mathbb{R}^3$, then

$\langle P, Q, R \rangle$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \ \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \text{ and } \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

$P_y = Q_x, \ Q_z = R_y, \ P_z = R_x$

6.11 Show that vector field $\mathbf{F}(x, y) = xy\mathbf{i} - x^2 y\mathbf{j}$ is not conservative.

$F = \langle P, Q \rangle$ If $P_y = Q_x$ then F is conservative

If F is conservative then there is a potential function f

$$\nabla f = F$$

$$P_y = \frac{\partial(xy)}{\partial y} = x \qquad Q_x = \frac{\partial(-x^2 y)}{\partial x} = -2xy$$

$P_y \neq Q_x$ F is not conservative.

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6.12 Is vector field $G(x, y, z) = \langle y, x, xyz \rangle$ conservative?

$\langle P, Q, R \rangle$

$P_y = Q_x \qquad 1 = 1$ ✓

$Q_z = R_y \qquad 0 = xz$

$P_z = R_x \qquad 0 = yz$

This vector field is <u>not</u> conservative.

$\nabla f = G$ is not possible

no potential function

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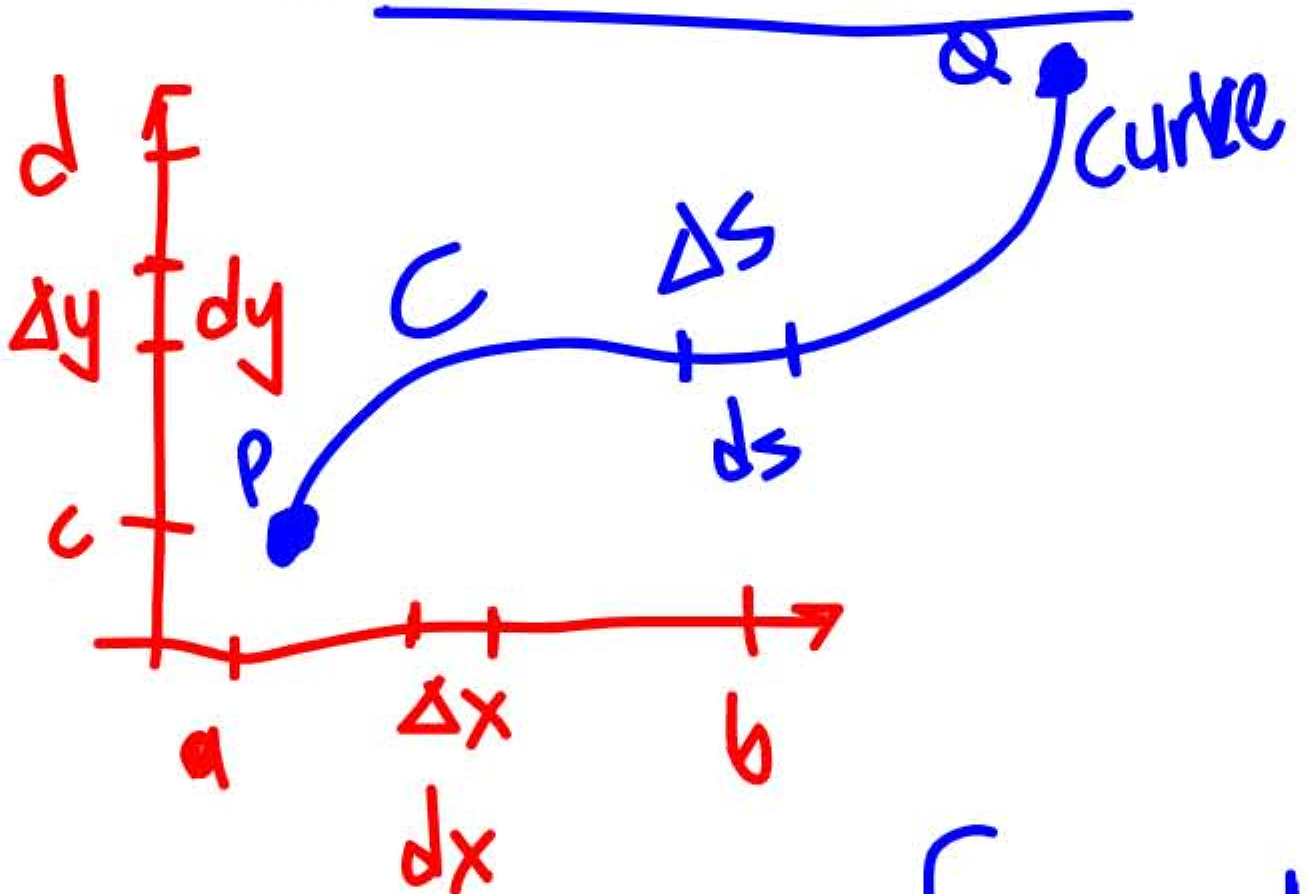
6.2 | Line Integrals



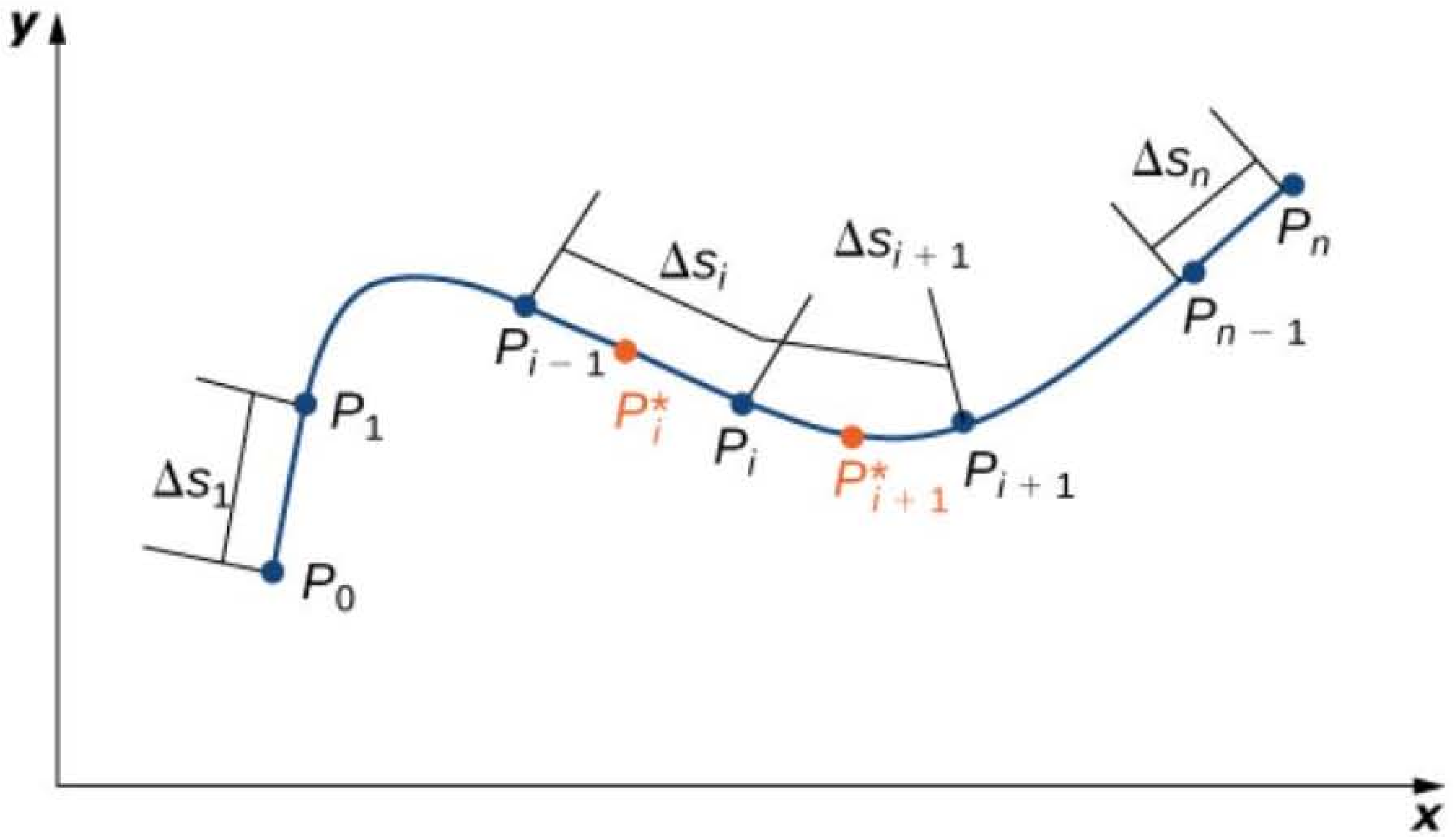


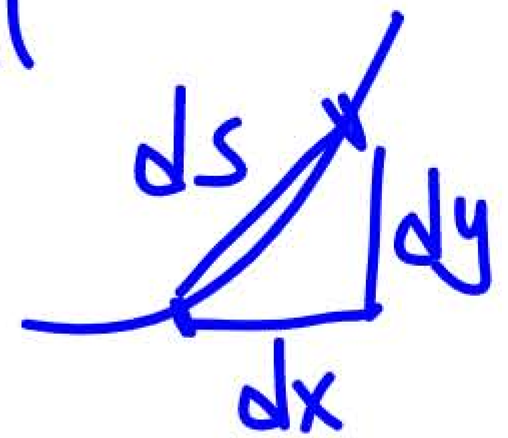
Figure 6.12 Curve C has been divided into n pieces, and a point inside each piece has been chosen.

$$\int_a^b f(x)\,dx$$

$$\int_c^d g(y)\,dy$$

$$\int_C f(x,y)\,ds$$

line integral



$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \sqrt{1 + (y')^2}\,dx$$

Definition

Let f be a function with a domain that includes the smooth curve C that is parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \le t \le b$. The **scalar line integral** of f along C is

C is in 3D

$$\int_C f(x, y, z)ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(P_i^*)\Delta s_i \tag{6.5}$$

if this limit exists (t_i^* and Δs_i are defined as in the previous paragraphs). If C is a planar curve, then C can be represented by the parametric equations $x = x(t),\ y = y(t),$ and $a \le t \le b$. If C is smooth and $f(x, y)$ is a function of two variables, then the scalar line integral of f along C is defined similarly as

C is in 2D.

$$\int_C f(x, y)ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(P_i^*)\Delta s_i,$$

if this limit exists.

geometric meaning of line integral, C is a curve on xy-plane.

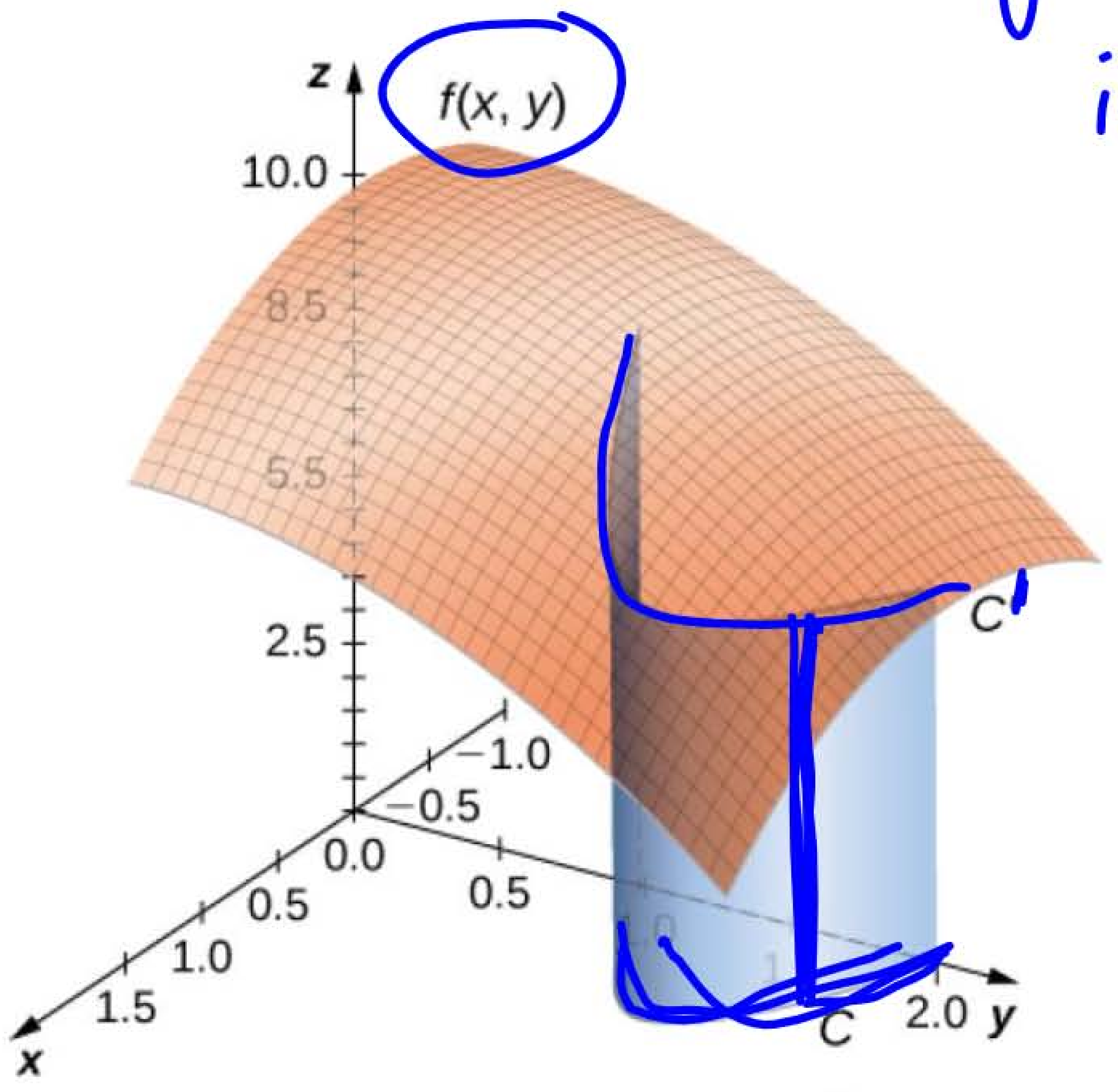
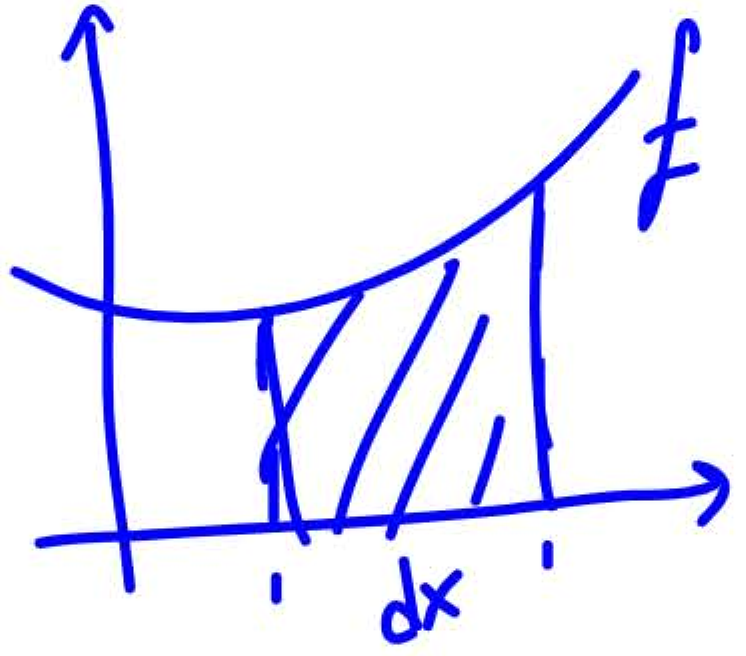


Figure 6.13 The area of the blue sheet is $\int_C f(x, y)\,ds$.



$\int_a^b f(x)dx = \text{Area}$

rectangular
small area

$f(x_i, y_i)\Delta s$

Theorem 6.3: Evaluating a Scalar Line Integral

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t)$, $a \leq t \leq b$. Then

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))\, \| \mathbf{r}'(t) \|\, dt. \tag{6.7}$$

Theorem 6.4: Scalar Line Integral Calculation

Let f be a continuous function with a domain that includes the smooth curve C with parameterization $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$. Then

$$\int_C f(x, y, z)ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt. \tag{6.8}$$

Similarly,

$r(t) = \langle x(t), y(t) \rangle$

$$\int_C f(x, y)ds = \int_a^b f(\mathbf{r}(t))\sqrt{(x'(t))^2 + (y'(t))^2}dt$$

if C is a planar curve and f is a function of two variables.

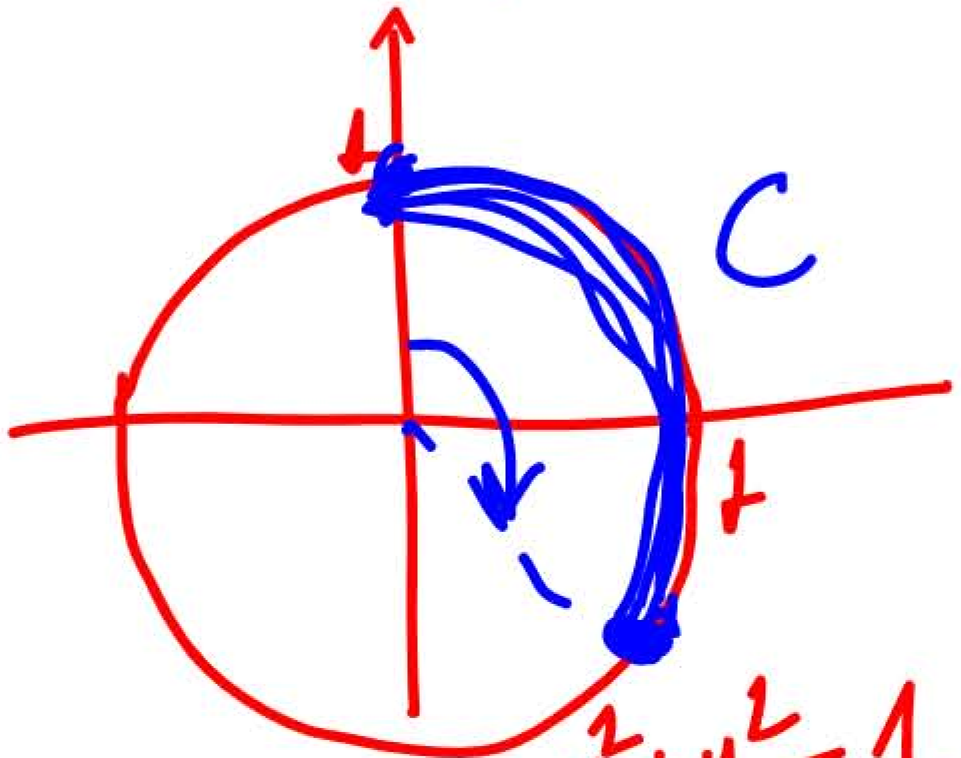
6.14 Evaluate $\int_C (x^2 + y^2 + z)\,ds$, where C is the curve with parameterization $\mathbf{r}(t) = \langle \sin(3t), \cos(3t) \rangle$, $0 \le t \le \frac{\pi}{4}$.

$r'(t) = \langle 3\cos 3t, -3\sin 3t \rangle$ $\quad \|r'(t)\| = 3$

$x(t) = \sin 3t$

$y(t) = \cos 3t$

$r(0) = \langle 0, 1 \rangle$



$x^2 + y^2 = 1$

$\sin^2(3t) + \cos^2(3t) =$

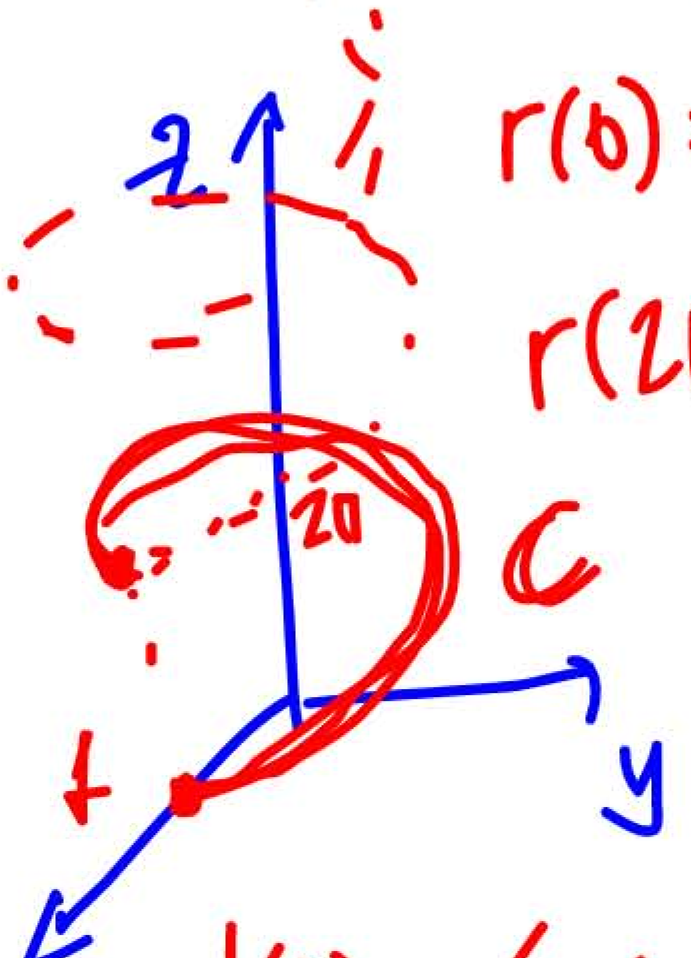
$$\int_0^{\pi/4} \left(\sin^2(3t) + \cos^2(3t) + z \right) 3\,dt$$

$$\int_0^{\pi/4} (1 + z)\, 3\,dt$$

Evaluating a Line Integral

Find the value of integral $\int_C (x^2 + y^2 + z)\,ds$, where C is part of the helix parameterized by

$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle, \quad 0 \le t \le 2\pi.$



$r(0) = \langle 1, 0, 0 \rangle$

$r(2\pi) = \langle 1, 0, 2\pi \rangle$

$r'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$\int_0^{2\pi} (\cos^2 t + \sin^2 t + t)\, \underbrace{\sqrt{2}\, dt}_{ds = \|r'(t)\|\, dt}$$

$$\int_0^{2\pi} (1+t)\sqrt{2}\, dt$$

$$= \left(\sqrt{2}\, t + \sqrt{2}\frac{t^2}{2}\right)\Big|_0^{2\pi}$$

$$= \sqrt{2}(2\pi + 2\pi^2) = 2\pi\sqrt{2}(1+\pi)$$