6 | VECTOR CALCULUS

6.4 | Green's Theorem

changing line integral into double integral

Learning Objectives

- **6.4.1** Apply the circulation form of Green's theorem.
- **6.4.2** Apply the flux form of Green's theorem.
- **6.4.3** Calculate circulation and flux on more general regions.

6.1 Vector fields

6.2 line integrals $\int_C f(x,y)\,ds$ or $\int_C F \cdot dr$

6.3 Conservative vector fields

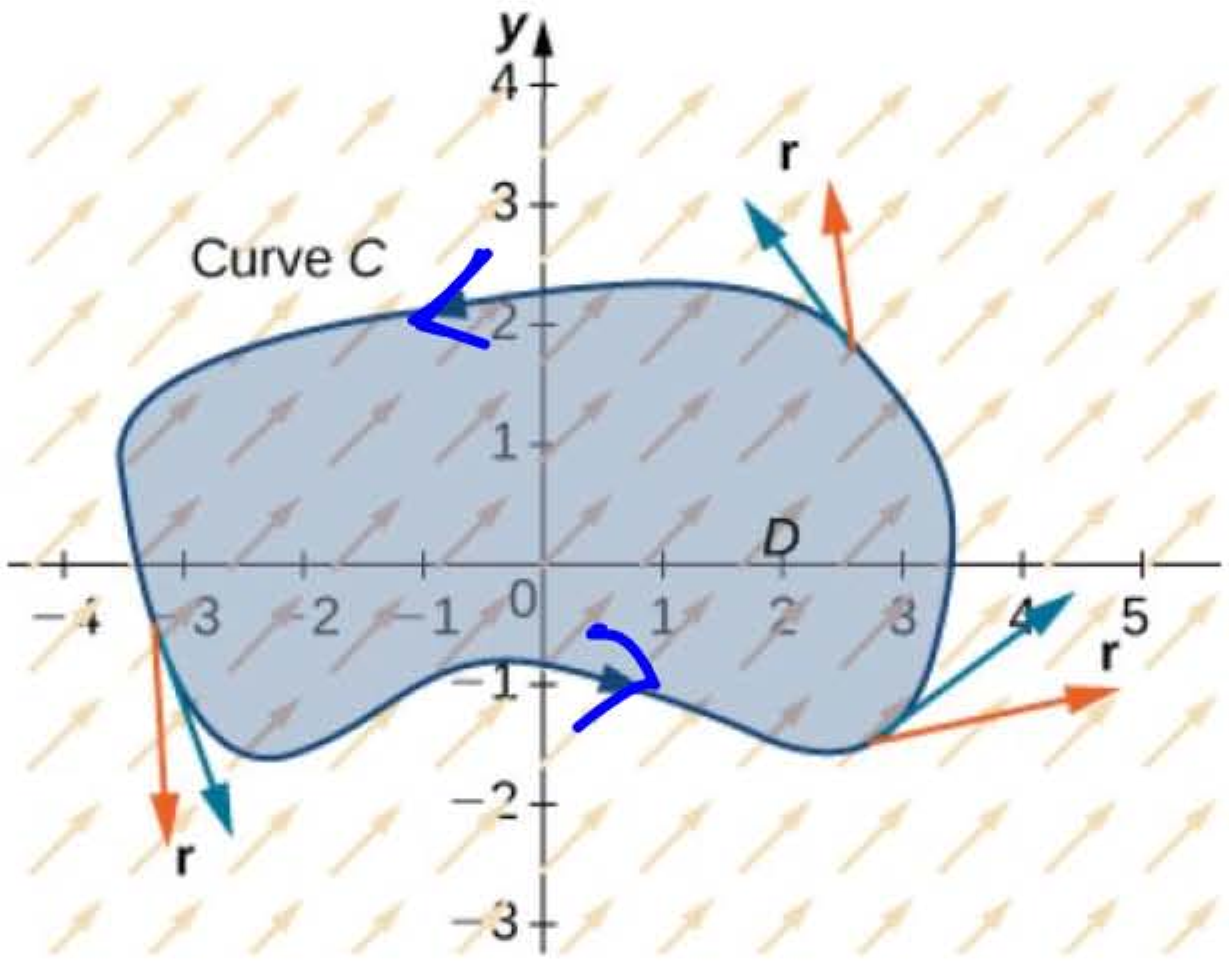
Extending the Fundamental Theorem of Calculus

Recall that the Fundamental Theorem of Calculus says that

$$\int_a^b f(x)dx = \int_a^b F'(x)dx = F(b) - F(a).$$

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

line integral



Theorem 6.12: Green's Theorem, Circulation Form

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve oriented counterclockwise (Figure 6.33). Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on D. Then,

$\langle P, Q \rangle \cdot \langle dx, dy \rangle$

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_D (Q_x - P_y) dA. \quad (6.13)$$

line int. double integral.

Applying Green's Theorem over a Rectangle

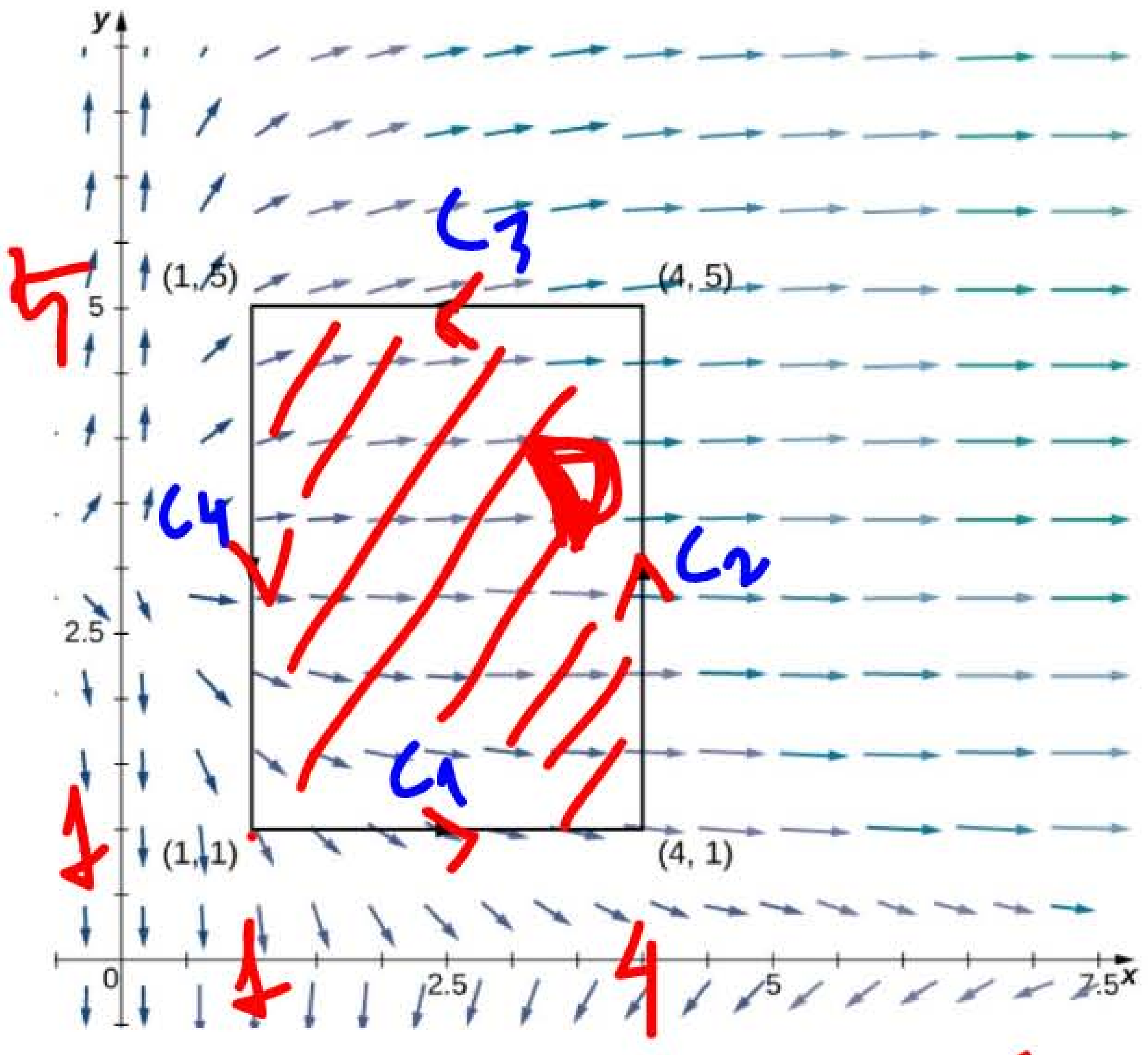
$F = \langle x^2y, y-3 \rangle$ — P, Q

We need to calculate line integrals over C_1, C_2, C_3, C_4 takes time.

Calculate the line integral

$$\oint_C x^2 y\,dx + (y-3)\,dy,$$

where C is a rectangle with vertices $(1, 1)$, $(4, 1)$, $(4, 5)$, and $(1, 5)$ oriented counterclockwise.



$1 \le x \le 4,\ 1 \le y \le 5$

$$\oint_C F\cdot dr = \iint_D (Q_x - P_y)\,dA \quad \text{Green's theorem}$$

$$\int_1^4\int_1^5 (0 - x^2)\,dy\,dx = \int_1^4 -x^2y\Big|_1^5\,dx$$

$$= \int_1^4 -4x^2\,dx = -\frac{4x^3}{3}\Big|_1^4 = -\frac{4}{3}(64-1) = -84$$

$$= -4\times 21$$

Applying Green's Theorem to Calculate Work

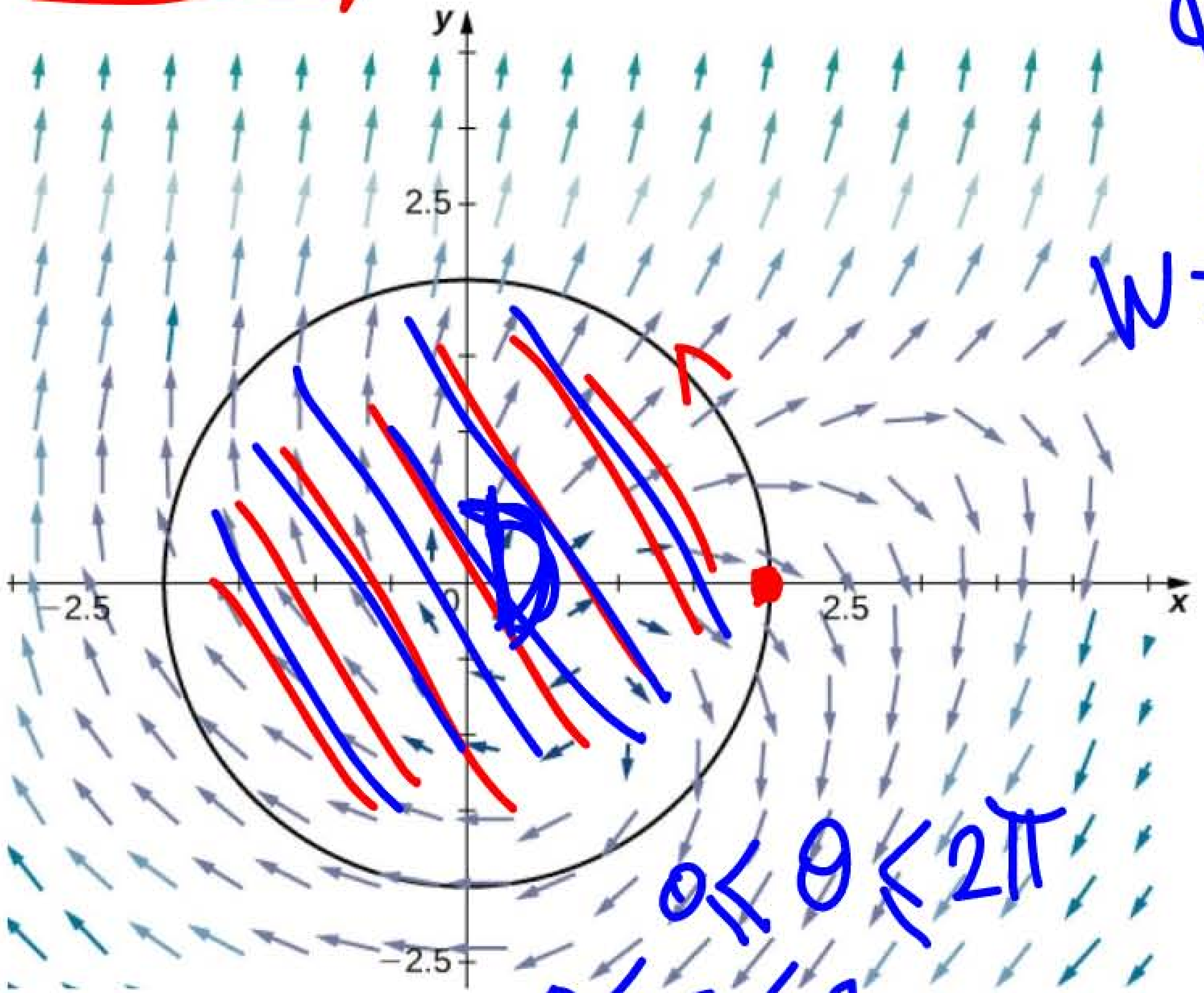
Calculate the work done on a particle by force field

$$\mathbf{F}(x, y) = \langle \underbrace{y + \sin x}_{P}, \underbrace{e^y - x}_{Q} \rangle$$

as the particle traverses circle $x^2 + y^2 = 4$ exactly once in the counterclockwise direction, starting and ending at point $(2, 0)$.

$$W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y)\, dA$$

Green's Thm



$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 2$

$Q_x = -1 \qquad P_y = 1$

area of D

$$W = \int_0^{2\pi}\int_0^{2} (-1-1)\, dA = -2 \iint_D 1\, dA$$

$$= -2\pi 2^2$$

$$= \underline{-8\pi}$$

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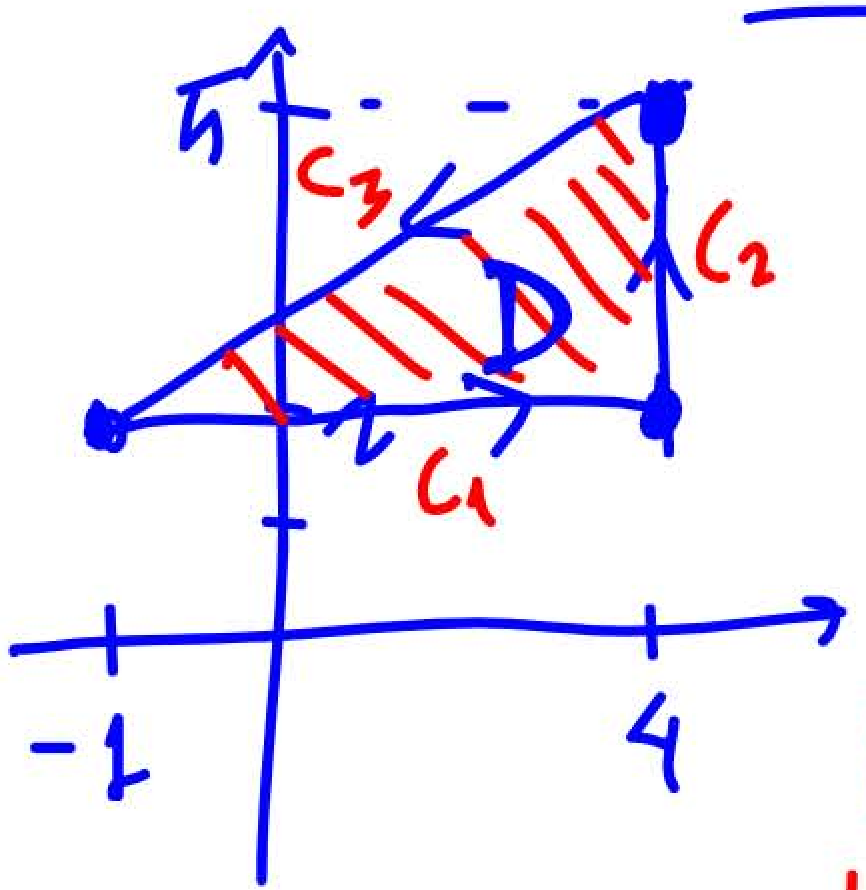
6.34 Use Green's theorem to calculate line integral

$$\oint_C \sin(x^2)dx + (3x - y)dy,$$

where C is a right triangle with vertices $(-1, 2)$, $(4, 2)$, and $(4, 5)$ oriented counterclockwise.

$$F = \langle \underbrace{\sin(x^2)}_{P}, \underbrace{3x - y}_{Q} \rangle$$



$$\oint_C F \cdot dr = \iint_D (Q_x - P_y)\, dA$$

$$= \int_{-1}^{4} \int_{2}^{\frac{3x}{5} + \frac{13}{5}} (3 - 0)\, dy\, dx$$

$$= \int_{-1}^{4} 3\left(\frac{3x}{5} + \frac{13}{5} - 2\right) dx$$

$$\int_{C_1} F \cdot dr + \int_{C_2} + \int_{C_3}$$

takes time.

D: $-1 \leq x \leq 4$

$2 \leq y \leq \frac{3x}{5} + \frac{13}{5}$

$$m_l = \frac{5-2}{4-(-1)} = \frac{3}{5}$$

$$y - 5 = \frac{3}{5}(x - 4), \quad y = \frac{3x}{5} + \frac{13}{5}$$

$$\int_{-1}^{4}\left(\frac{9x}{5}+\frac{9}{5}\right)dx = \frac{9}{5}\left(\frac{x^2}{2}+x\right)\Big|_{-1}^{4}$$

$$= \frac{9}{5}\left[8+4-\left(\frac{1}{2}-1\right)\right] = \frac{9}{5}\,\frac{25}{2} = \frac{45}{2}$$

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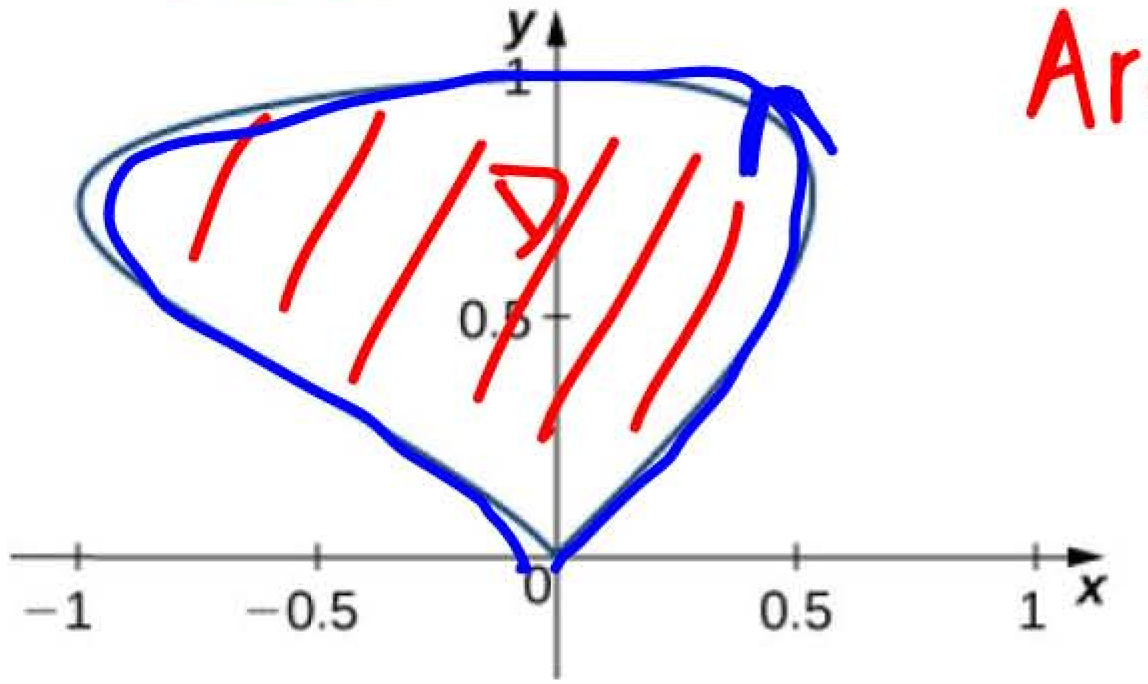
6.35 Find the area of the region enclosed by the curve with parameterization $\mathbf{r}(t) = \langle \sin t \cos t, \sin t \rangle$, $0 \le t \le \pi$.



$$\text{Area} = \iint_D 1 \cdot dA = \oint_C F \cdot dr$$

$$\boxed{1 = Q_x - P_y}$$

$$F = \langle P, Q \rangle$$

$$Q = \frac{x}{2} \qquad P = -\frac{y}{2} \qquad \boxed{F = \left\langle \frac{x}{2}, -\frac{y}{2} \right\rangle}$$

Double integral is not easy to calculate we will calculate the corresspponding line integral.

$$x(t) = \sin t \cos t = \frac{\sin 2t}{2} \qquad dx = \cos 2t \, dt$$

$$y(t) = \sin t \qquad dy = \cos t \, dt$$

$$\oint_C F \cdot dr = \int_0^{\pi} \left\langle \frac{x}{2}, -\frac{y}{2} \right\rangle \cdot \langle \cos 2t, \cos t \rangle \, dt$$

$x = \frac{\sin 2t}{2}$

$y = \sin t$

$$= \int_0^{\pi} \left(\frac{\sin 2t \cos 2t}{4} - \frac{\sin t \cos t}{2} \right) dt$$

$\sin t \cos t = \frac{\sin 2t}{2}$

double angle.

$$= \int_0^{\pi} \left(\frac{\sin 4t}{8} - \frac{\sin 2t}{4} \right) dt = \left(-\frac{\cos 4t}{32} + \frac{\cos 2t}{8} \right) \Big|_0^{\pi}$$

$$= \frac{1}{8} \left[\left(-\frac{1}{4} + 1 \right) - \left(\frac{1}{4} + 1 \right) \right] = \frac{1}{8} \left(\frac{3}{4} - \left(-\frac{3}{4} \right) \right) = \frac{3}{16}$$

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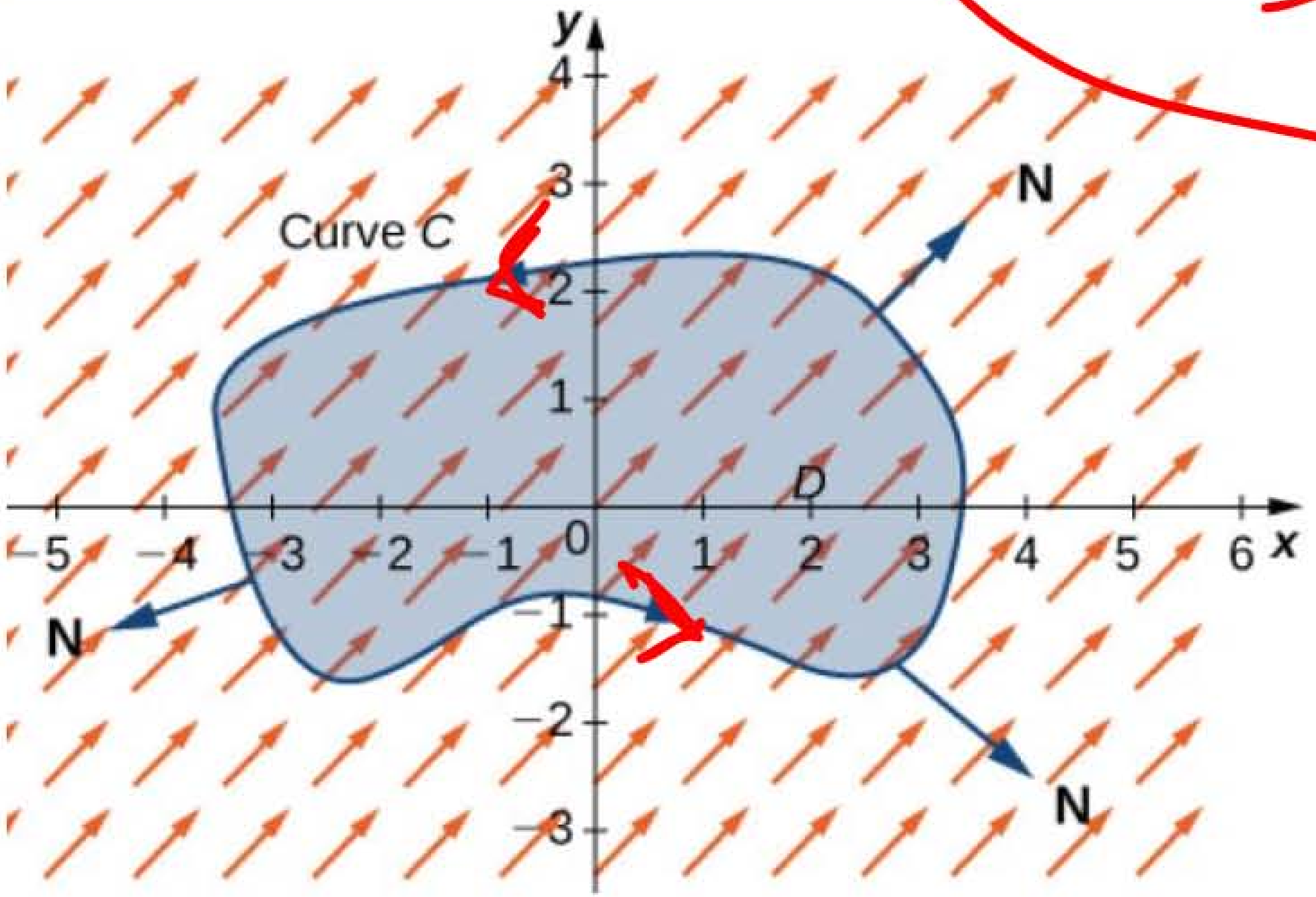
Flux Form of Green's Theorem

Theorem 6.13: Green's Theorem, Flux Form

Let D be an open, simply connected region with a boundary curve C that is a piecewise smooth, simple closed curve that is oriented counterclockwise (Figure 6.38). Let $\mathbf{F} = \langle P, Q \rangle$ be a vector field with component functions that have continuous partial derivatives on an open region containing D. Then,

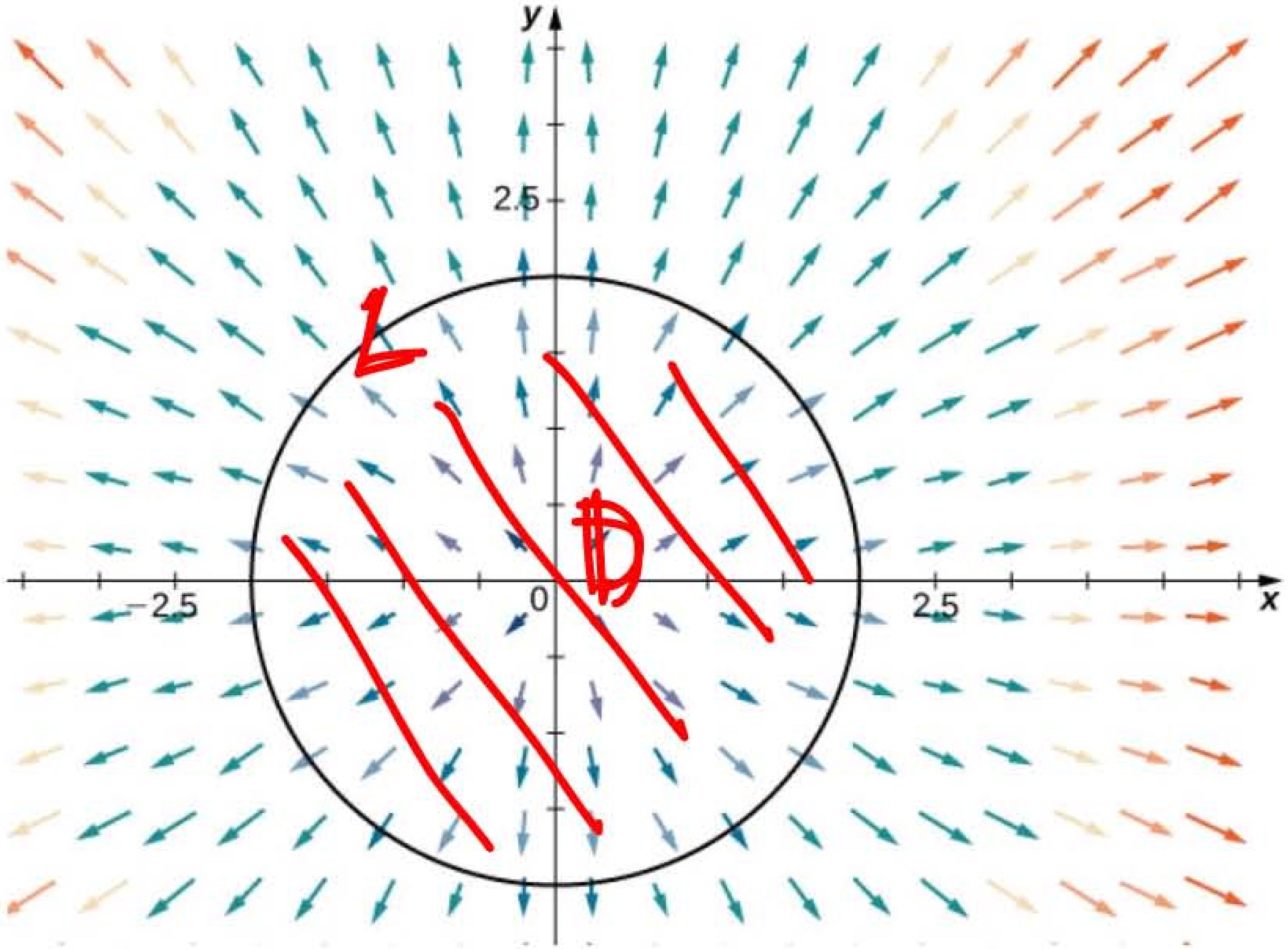
$$\oint_C \mathbf{F} \cdot \mathbf{N} ds = \iint_D P_x + Q_y \, dA. \quad (6.15)$$



Applying Green's Theorem for Flux across a Circle

$P_x = 1 \quad Q_y = 1$

Let C be a circle of radius r centered at the origin (**Figure 6.39**) and let $\mathbf{F}(x, y) = \langle \overset{P}{x}, \overset{Q}{y} \rangle$. Calculate the flux across C.



$$\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D (1+1)\, dA$$

$$= 2 \iint_D dA = 2 \times \text{Area of } D$$

$$= 2\pi r^2$$

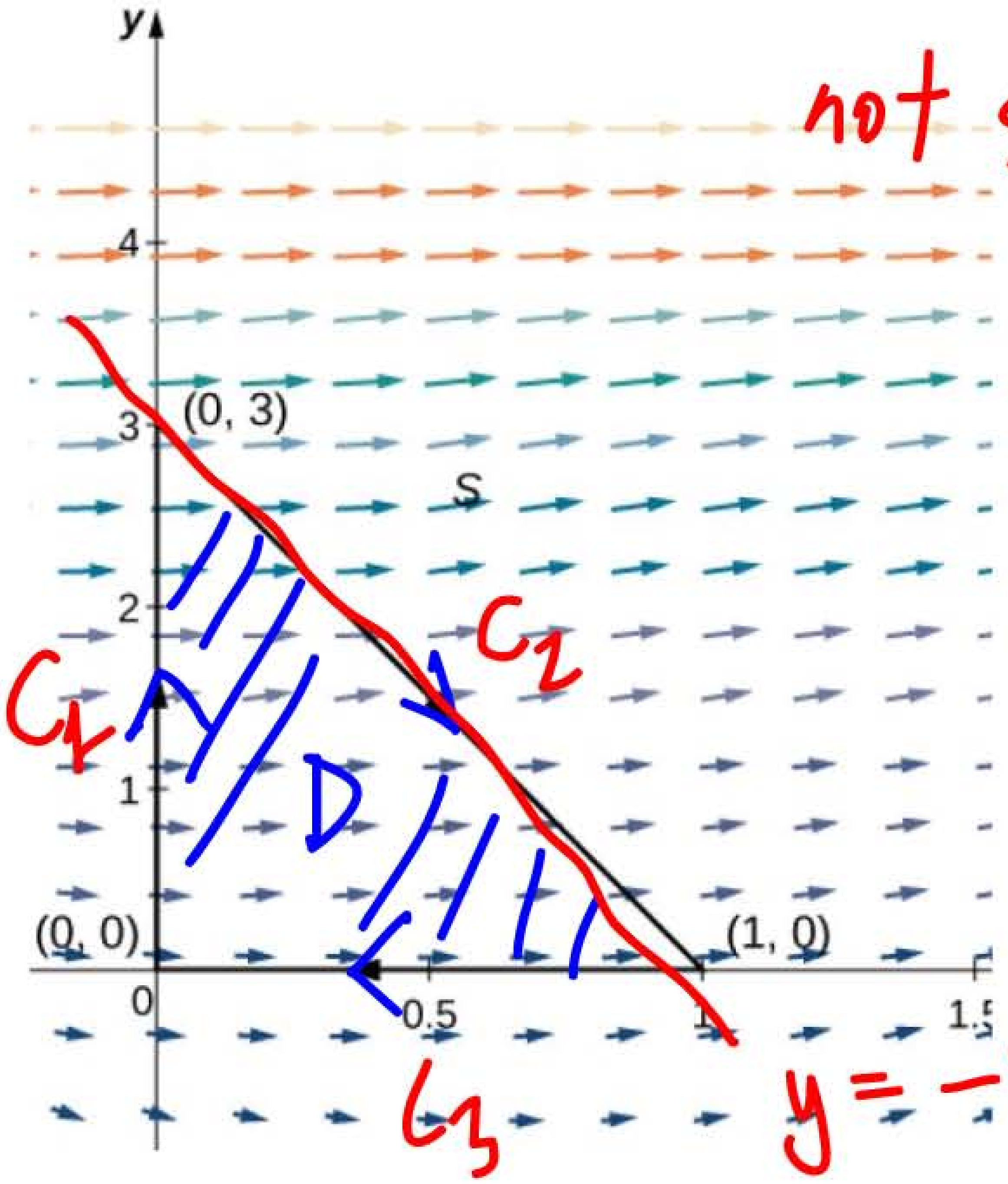
Applying Green's Theorem for Flux across a Triangle

Let S be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 3)$ oriented clockwise (**Figure 6.40**). Calculate the flux of $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle x^2 + e^y, x + y \rangle$ across S.

Green's T.

not short

$$\oint_C F\cdot N\,ds = -\iint_D (P_x + Q_y)\,dA$$



$y = -3x+3$

D: $0 \le x \le 1$

$0 \le y \le -3x+3$

$P_x = 2x$

$Q_y = 1$

$$\int_0^1 \int_0^{-3x+3} (2x+1)\,\cancel{dy}\,dx$$

$$(2x+1)y\Big|_0^{-3x+3}$$

$$\int_0^1 (2x+1)y\Big|_0^{-3x+3} dx = \int_0^1 (2x+1)(-3x+3)\,dx$$

$$= \int_0^1 (-6x^2+3x+3)\,dx = \left(-2x^3+\frac{3x^2}{2}+3x\right)\Bigg|_0^1$$

$$= -2+\frac{3}{2}+3 = \frac{5}{2}$$

C is clockwise

$$\oint_C F\cdot N\,ds = -\frac{5}{2}$$

6.36 Calculate the flux of $\mathbf{F}(x, y) = \langle x^3, y^3 \rangle$ across a unit circle oriented counterclockwise.

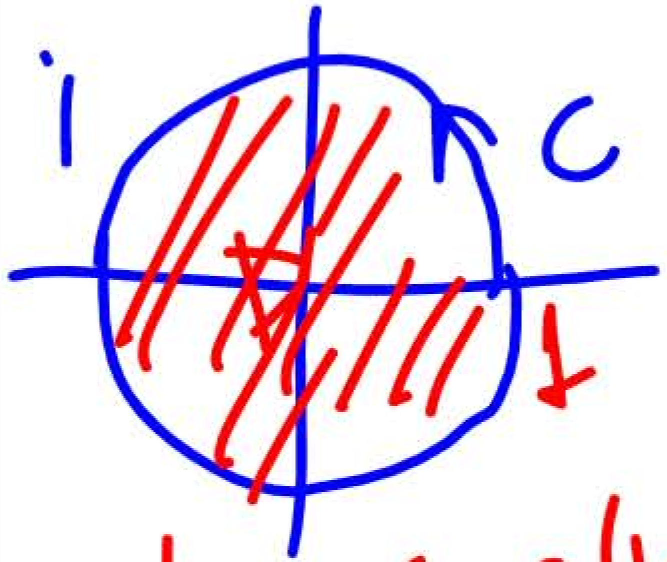
$$\oint_C F \cdot N\, ds = \iint_D (P_x + Q_y)\, dA$$

$P = x^3$

$Q = y^3$

$$\iint_D (3x^2 + 3y^2)\, dA$$

not easy in (x,y) because of D



polar coordinates

$x^2 + y^2 = r^2$

$\boxed{dA = dx\,dy}$

$= r\, dr\, d\theta$

D: $0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi}\int_0^1 3r^2 \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} 3\frac{r^4}{4}\Big|_0^1\, d\theta = \frac{3\theta}{4}\Big|_0^{2\pi} = \frac{3\pi}{2} = \text{flux}$$

similar to conservative field.

The following statements are all equivalent ways of defining a source-free field $\mathbf{F} = \langle P, Q \rangle$ on a simply connected domain (note the similarities with properties of conservative vector fields):

1. The flux $\oint_C \mathbf{F} \cdot \mathbf{N} ds$ across any closed curve C is zero.
2. If C_1 and C_2 are curves in the domain of $\mathbf{F}$ with the same starting points and endpoints, then $\int_{C_1} \mathbf{F} \cdot \mathbf{N} ds = \int_{C_2} \mathbf{F} \cdot \mathbf{N} ds$. In other words, flux is independent of path.
3. There is a **stream function** $g(x, y)$ for $\mathbf{F}$. A stream function for $\mathbf{F} = \langle P, Q \rangle$ is a function g such that $P = g_y$ and $Q = -g_x$. Geometrically, $\mathbf{F} = (a, b)$ is tangential to the level curve of g at (a, b). Since the gradient of g is perpendicular to the level curve of g at (a, b), stream function g has the property $\boxed{\mathbf{F}(a, b) \bullet \nabla g(a, b) = 0}$ for any point (a, b) in the domain of g. (Stream functions play the same role for source-free fields that potential functions play for conservative fields.)
4. $P_x + Q_y = 0$ ✓

F is source-free

$F \cdot \nabla g = yx - xy = 0$

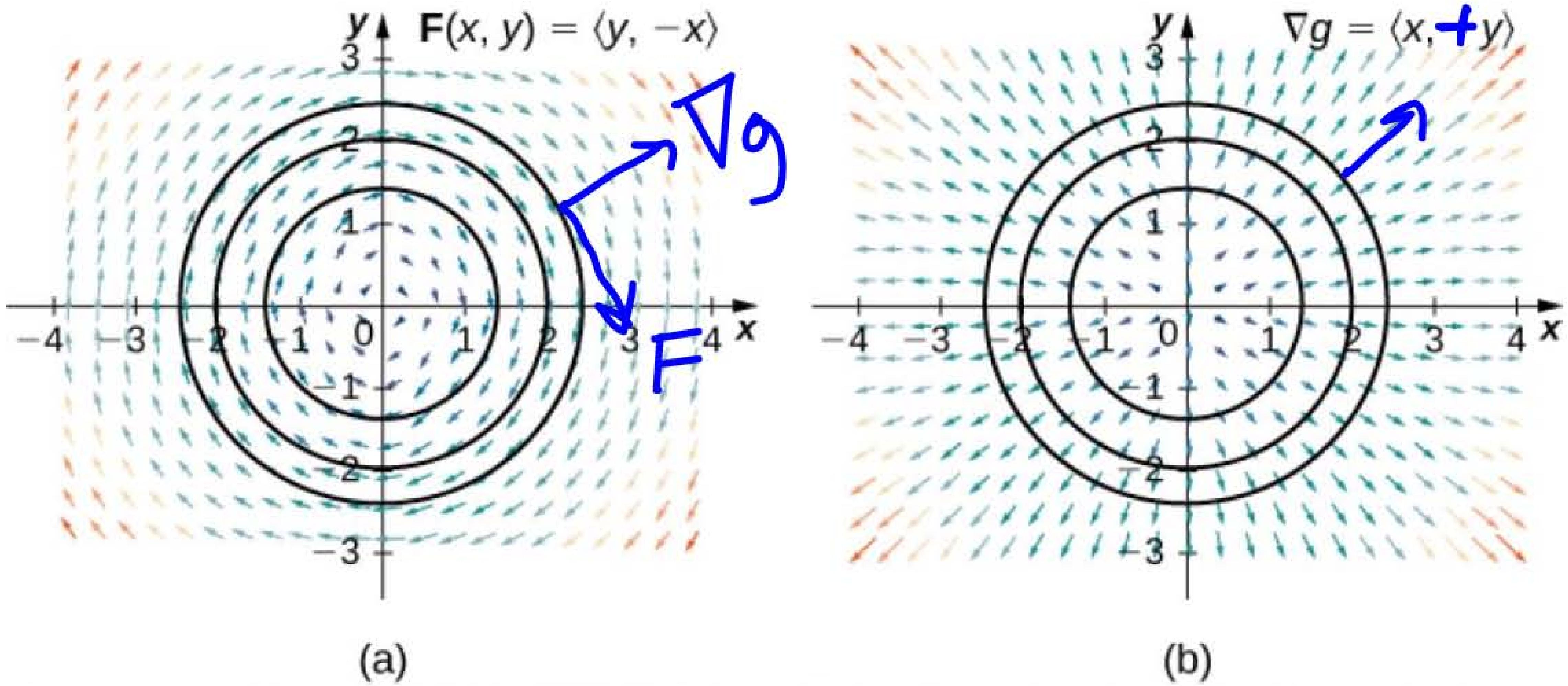


Figure 6.42 (a) In this image, we see the three-level curves of g and vector field $\mathbf{F}$. Note that the $\mathbf{F}$ vectors on a given level curve are tangent to the level curve. (b) In this image, we see the three-level curves of g and vector field ∇g. The gradient vectors are perpendicular to the corresponding level curve. Therefore, $\underbrace{\mathbf{F}(a, b) \bullet \nabla g(a, b) = 0}$ for any point in the domain of g.

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6.37 Find a stream function for vector field $\mathbf{F}(x, y) = \langle x \sin y, \cos y \rangle$.

$\langle P, Q \rangle \cdot \langle Q, -P \rangle = 0$

Find $g(x,y)$ such that $F \cdot \nabla g = 0$.

$\nabla g = \langle \cos y, -x \sin y \rangle$ find $g(x,y)$

integrate $\cos y$ with respect to x

$x \cos y + h(y)$

differentiate with respect to y

$-x \sin y + h'(y) = -x \sin y$

$h'(y) = 0$, $h(y) = C$.

stream function is

$g(x,y) = x \cos y + C$.

Satisfying Laplace's Equation

Laplace's equation $f_{xx} + f_{yy} = 0.$ function that satisfies Laplace's equation is called a *harmonic* function.

For vector field $\mathbf{F}(x, y) = \langle e^x \sin y, e^x \cos y \rangle$, verify that the field is both conservative and source free, find a potential function for **F**, and verify that the potential function is harmonic.

$F = \langle P, Q \rangle$ $P_y = Q_x$ implies that F is conservative.

$e^x \cos y = e^x \cos y$ ✓ it is conservative.

$P_x + Q_y = 0$ implies F is source-free.

$e^x \sin y + (-e^x \sin y) = 0$ ✓ F is source free.

$f(x,y) = e^x \sin y + C$ is a potential functio

$e^x \sin y + h(y)$

$e^x \cos y + h'(y) = e^x \cos y$ $\nabla f = F$

$h'(y) = 0$, $h(y) = C$

$P_x + Q_y = (f_x)_x + (f_y)_y = 0$

$f_{xx} + f_{yy} = 0$

We may also check it directly

Slide 21 100%

6.38 Is the function $f(x, y) = e^{x+5y}$ harmonic?

$f_x = e^{x+5y}$

$f_{xx} = e^{x+5y}$

$f_y = 5e^{x+5y}$

$f_{yy} = 25e^{x+5y}$

$f_{xx} + f_{yy} = 0$

$f_{xx} + f_{yy} = 26e^{x+5y} \neq 0$

the function does not satisfy Laplace's equation

So it is <u>not</u> harmonic.

Green's Theorem on General Regions

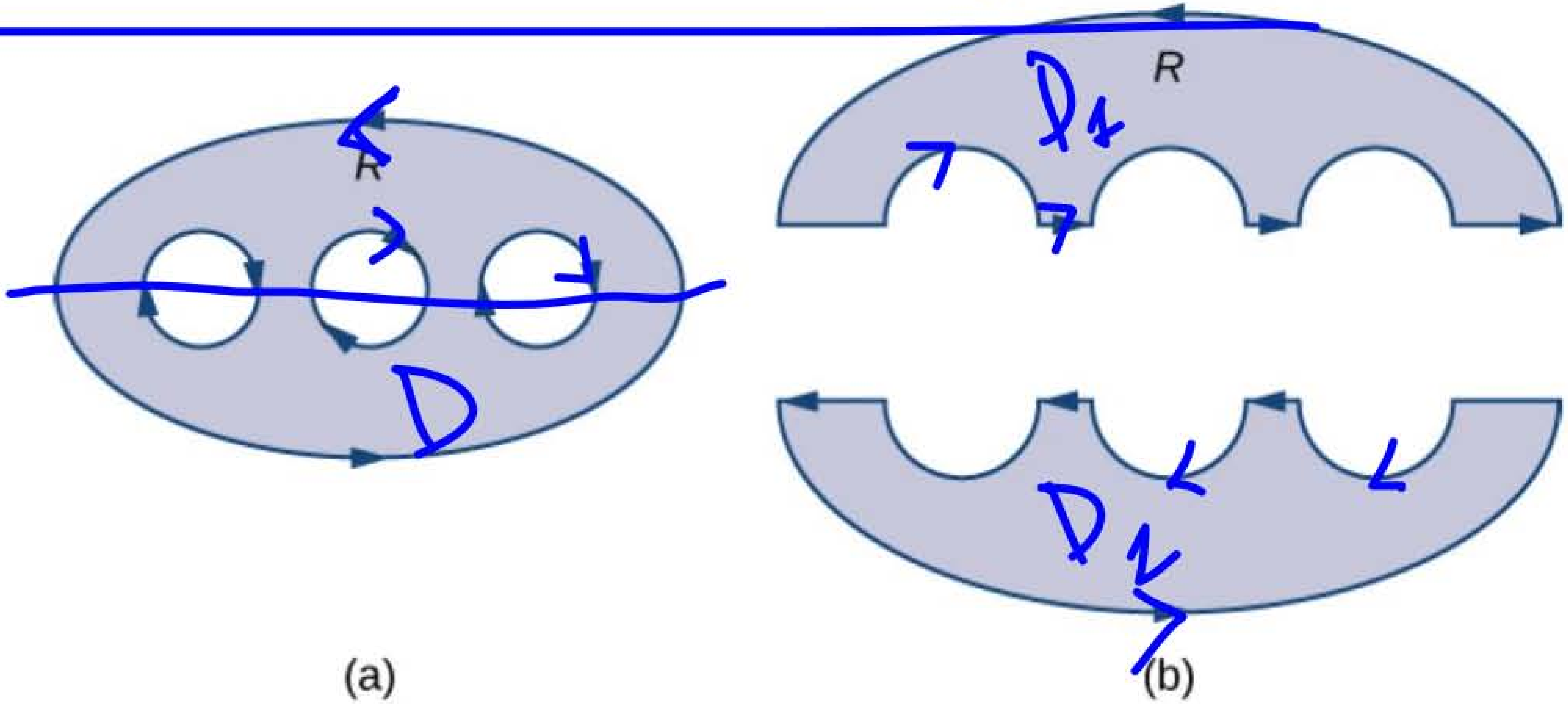


Figure 6.44 (a) Region D with an oriented boundary has three holes. (b) Region D split into two simply connected regions has no holes.

If $\mathbf{F}$ is a vector field defined on D, then Green's theorem says that

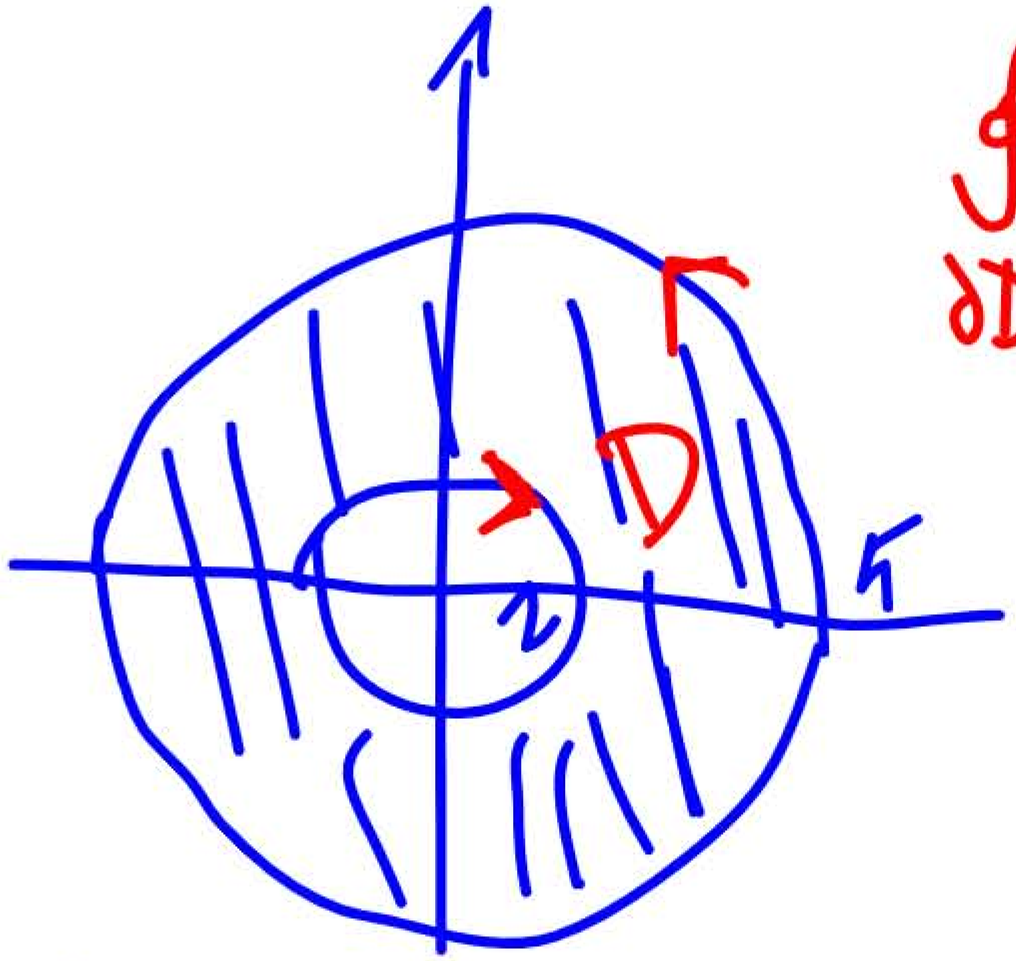
$$\begin{aligned}\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \oint_{\partial D_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{\partial D_2} \mathbf{F} \cdot d\mathbf{r} \\ &= \iint_{D_1} Q_x - P_y \, dA + \iint_{D_2} Q_x - P_y \, dA \\ &= \iint_D (Q_x - P_y) dA.\end{aligned}$$

6.39 Calculate integral $\oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}$, where D is the annulus given by the polar inequalities $2 \le r \le 5$, $0 \le \theta \le 2\pi$, and $\mathbf{F}(x, y) = \langle x^3, 5x + e^y \sin y \rangle$.

$P = x^3 \quad P_y = 0$

$Q = 5x + e^y \sin y \quad Q_x = 5$

$$\oint_{\partial D} F \cdot dr = \iint_D (Q_x - P_y)\, dA$$



$D.\ 2 \leq r \leq 5$

$0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi}\int_2^5 5\, r\, dr\, d\theta = \int_0^{2\pi} \frac{5r^2}{2}\Big|_2^5\, d\theta$$

$$= \int_0^{2\pi} \frac{5}{2}(5^2 - 2^2)\, d\theta = \frac{105}{2}\theta \Big|_0^{2\pi} = 105\pi$$