1. Evaluate the integral $\iint_D (x^2 - 2y + xy)\, dA$ where $D = [-1, 1] \times [0, 2]$. $-1 \leq x \leq 1,\ 0 \leq y \leq 2$

$$\int_{-1}^{1}\int_{0}^{2}(x^2-2y+xy)\,dy\,dx = \int_{-1}^{1}\left(x^2y-y^2+x\frac{y^2}{2}\right)\Big|_0^2 dx = \int_{-1}^{1}(2x^2-4+2x)\,dx$$

$$=\left(\frac{2x^3}{3}-4x+x^2\right)\Big|_{-1}^{1} = \left(\frac{2}{3}-4+1\right)-\left(-\frac{2}{3}+4+1\right) = \frac{4}{3}-8 = -\frac{20}{3}$$

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2. Evaluate the integral $\iint_R (2xy + e^x)\, dA$ where $R = [0, 1] \times [0, 1]$.

$$\int_0^1\int_0^1 (2xy+e^x)\,dy\,dx = \int_0^1 (xy^2+ye^x)\Big|_0^1 dx = \int_0^1 (x+e^x)\,dx = \left(\frac{x^2}{2}+e^x\right)\Big|_0^1$$

$$=\frac{1}{2}+e-1 = e-\frac{1}{2}$$

---

3. Evaluate the integral $\iint_D (2x^2y^{-2} + 2y)\, dA$ where $D = \{(x, y)\,|\,1 \leq x \leq 2,\ 1 \leq y \leq x\}$.

$$\int_1^2\int_1^x (2x^2y^{-2}+2y)\,dy\,dx = \int_1^2 \left(\frac{2x^2y^{-1}}{-1}+y^2\right)\Big|_1^x dx = \int_1^2 \left(\left(-\frac{2x^2}{x}+x^2\right)-(-2x^2+1)\right)dx$$

$$=\int_1^2 (3x^2-2x-1)\,dx = (x^3-x^2-x)\Big|_1^2 = (8-4-2)-(1-1-1) = 2-(-1) = 3$$

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4. Evaluate the integral $\iint_D (1 + \cos x)\, dA$ where $D = \{(x, y)\,|\,0 \leq x \leq \pi,\ 0 \leq y \leq \sin x\}$.

$$\int_0^\pi\int_0^{\sin x}(1+\cos x)\,dy\,dx = \int_0^\pi (1+\cos x)y\Big|_0^{\sin x} dx = \int_0^\pi (1+\cos x)\sin x\,dx$$

$$=\int_0^\pi (\sin x+\cos x\sin x)\,dx = \int_0^\pi \left(\sin x+\frac{\sin 2x}{2}\right)dx = \left(-\cos x-\frac{\cos 2x}{4}\right)\Big|_0^\pi = -(-1)-\frac{1}{4}-\left(-1-\frac{1}{4}\right)$$

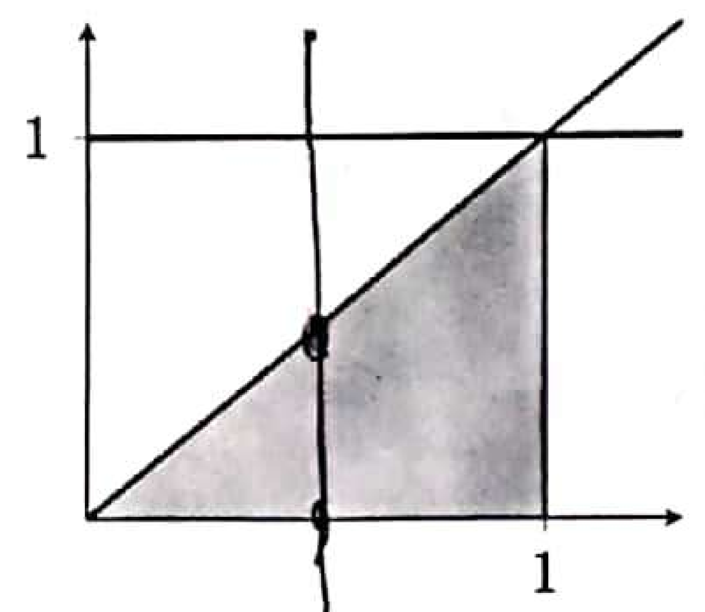
$$=2$$

---

5. Find the volume between the surface $z = x + xy$ and $D = [0, 2] \times [0, 1]$ on xy-plane.

$$V=\int_0^2\int_0^1 (x+xy)\,dy\,dx = \int_0^2 \left(xy+\frac{xy^2}{2}\right)\Big|_0^1 dx = \int_0^2 \frac{3x}{2}\,dx = \frac{3x^2}{4}\Big|_0^2 = 12,$$

6. Find the volume between the surface $z = e^{-x^2}$ and D on xy-plane, where D is as shown below:

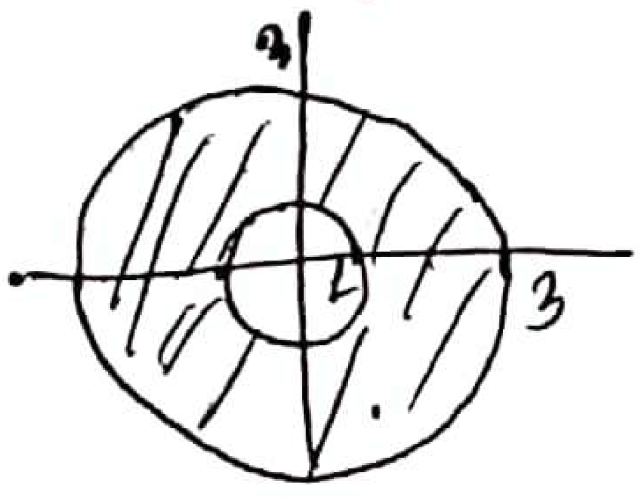


$V = \int_0^1\int_0^x e^{-x^2}\,dy\,dx = \int_0^1 y e^{-x^2}\Big|_0^x dx = \int_0^1 x e^{-x^2}\,dx$ $\quad u=-x^2$, $du=-2x\,dx$

$0\leq x\leq 1,\ 0\leq y\leq x$

$\int_0^{-1} e^u\left(-\frac{du}{2}\right) = -\frac{1}{2}(e^{-1}-e^0) = -\frac{1}{2}\left(\frac{1}{e}-1\right)$
$= (e-1)/2e$

7. Evaluate the integral $\int\int_D (x+y)\,dA$ where D is the region between the circles $x^2+y^2=1$ and $x^2+y^2=9$.



$\int_0^{2\pi}\int_1^3 r(\cos\theta+\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}(\cos\theta+\sin\theta)\frac{r^3}{3}\Big|_1^3\,d\theta$

$\int_0^{2\pi}\frac{26}{3}(\cos\theta+\sin\theta)\,d\theta = \frac{26}{3}(\sin\theta-\cos\theta)\Big|_0^{2\pi} = 0$

$0\leq\theta\leq 2\pi,\ 1\leq r\leq 3$

$= \frac{26}{3}(-1-(-1))$

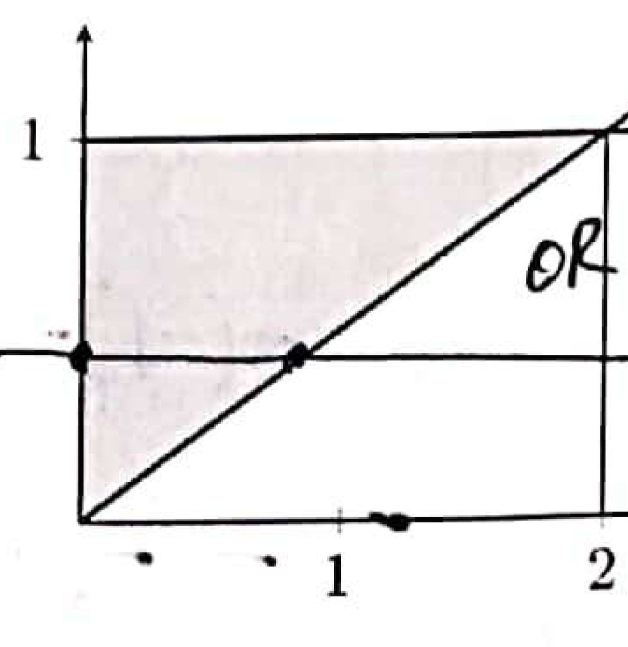
8. Evaluate the integral $\int\int_D \frac{1}{x^2+y^2}\,dA$ where D is the region between the circles $\underbrace{x^2+y^2=1}_{r=1}$ and $\underbrace{x^2+y^2=e^2}_{r=e}$.

$0\leq\theta\leq 2\pi$
$1\leq r\leq e$

$\int_0^{2\pi}\int_1^e \frac{1}{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\int_1^e \frac{1}{r}\,dr\,d\theta = \int_0^{2\pi}\ln r\Big|_1^e\,d\theta$

$= \int_0^{2\pi}(\ln e - \ln 1)\,d\theta = \int_0^{2\pi}1\,d\theta = \theta\Big|_0^{2\pi} = 2\pi$

9. Evaluate the integral $\int_0^2\int_{x/2}^1 e^{y^2}\,dy\,dx$ by switching the order of integration to $dxdy$.

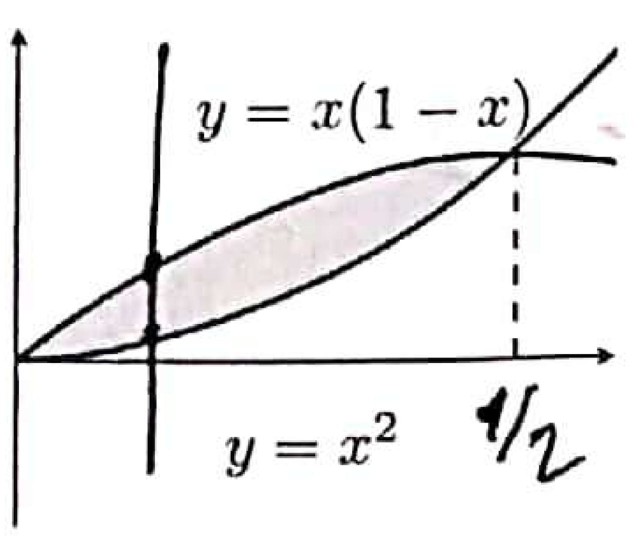


$\int_0^2\int_{x/2}^1 e^{y^2}\,dy\,dx = \int_0^1\int_0^{2y} e^{y^2}\,dx\,dy$ $\quad u=y^2$, $du=2y\,dy$

$= \int_0^1 x e^{y^2}\Big|_0^{2y}\,dy = \int_0^1 2y e^{y^2}\,dy = \int_0^1 e^u\,du = e^u\Big|_0^1$
$= e-1$

$0\leq y\leq 1,\ 0\leq x\leq 2y$

10. Evaluate $\int\int_D x\,dy\,dx$ where D is as below;



$x-x^2=x^2$
$0=2x^2-x$
$0=x(2x-1)$
$x=0,\ x=1/2$

$\int_0^{1/2}\int_{x^2}^{x-x^2} x\,dy\,dx = \int_0^{1/2} xy\Big|_{x^2}^{x-x^2}\,dx$

$= \int_0^{1/2} x((x-x^2)-x^2)\,dx = \int_0^{1/2}(x^2-2x^3)\,dx$

$0\leq x\leq 1/2,\ x^2\leq y\leq x-x^2$

$= \left(\frac{x^3}{3}-\frac{x^4}{2}\right)\Big|_0^{1/2} = \frac{1}{24}-\frac{1}{32} = \frac{4-3}{96} = \frac{1}{96}$

11. Determine whether the vector field $F(x, y) = \langle \cos x - 2xy, e^y - x^2 \rangle$ is conservative. Find a potential function if it is conservative.

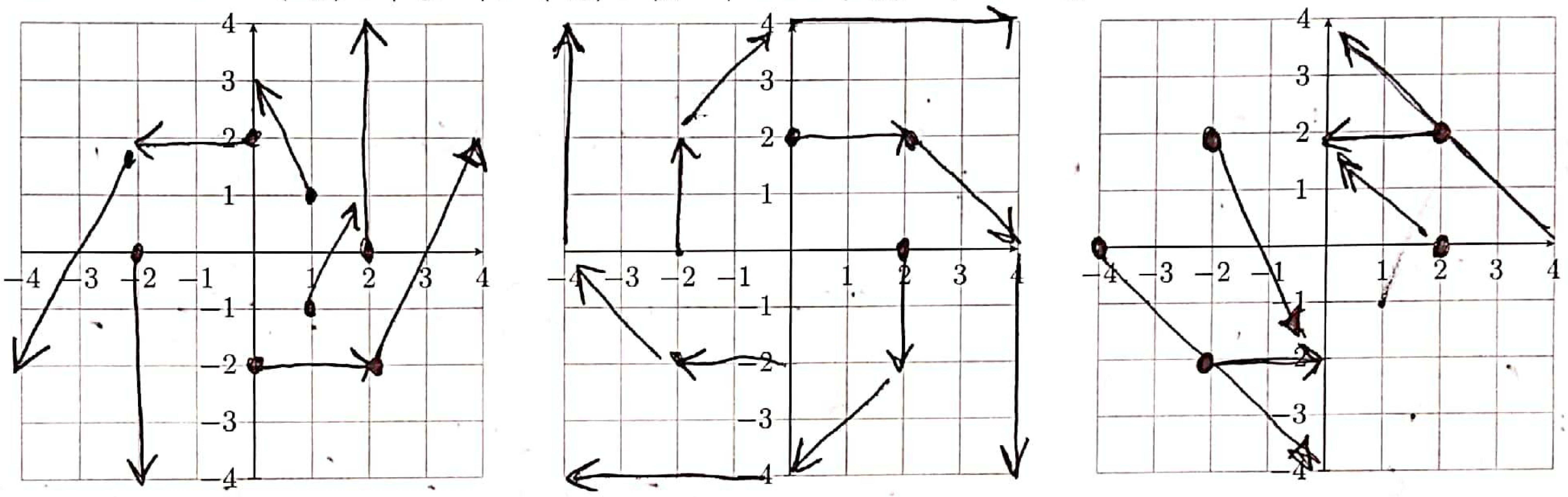
$M = \cos x - 2xy \quad N_x = -2x \quad N_x = M_y$ therefore it is conservative.

$N = e^y - x^2 \quad M_y = -2x \quad f(x,y) = e^y - x^2 y + \sin x$

is a potential function.

$\nabla f = F$

12. Sketch some vectors $F(x, y) = \langle -y, 2x \rangle$, $G(x, y) = \langle y, -x \rangle$ and $H(x, y) = \langle -x, x - y \rangle$



counter clockwise

Clockwise

13. Find the divergences div F, div G and div H for the vector fields in the previous question.

$\text{div } F = M_x + N_y = 0 + 0 = 0$

$\text{div } G = 0 + 0 = 0$

$\text{div } H = -1 + (-1) = -2$, sink., y-axis.

14. Find the curl F for the vector field $F(x, y, z) = x^2 y i + yz j - xyz k$ and evaluate the curl at the point $P(1, -1, 2)$.

$M = x^2 y$

$N = yz$

$P = -xyz$

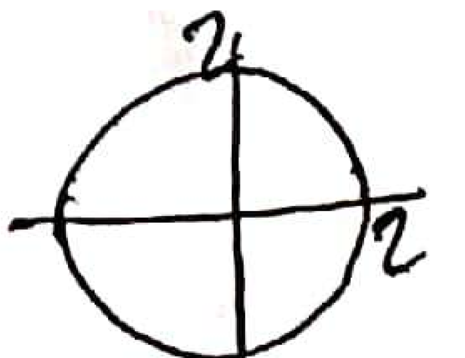
$\text{Curl } F(x,y,z) = (P_y - N_z)i - (P_x - M_z)j + (N_x - M_y)k$

$= (-xz - y)i - (-yz - 0)j + (0 - x^2)k$

$\text{Curl } F(1,-1,2) = (-2+1)i + (-1\times 2)j - 1^2 k$

$= -i - 2j - k = \langle -1, -2, -1 \rangle$

15. Use Green's theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \langle y^3, x^3+3x^2y\rangle$ and C is the circle $x^2+y^2=4$, oriented counter clockwise.



$N_x = 3x^2+6xy$

$M_y = 3y^2$

$$\iint_D (N_x - M_y)\,dA = \int_0^{2\pi}\int_0^2 \left(3r^2\cos^2\theta + 6r^2\cos\theta\sin\theta + 3r^2\sin^2\theta\right) r\,dr\,d\theta.$$

D: $0\leqslant r\leqslant 2$, $0\leqslant\theta\leqslant 2\pi$

$x = r\cos\theta$, $y = r\sin\theta$

$$= \int_0^{2\pi}\int_0^2 3r^3(1+3\sin 2\theta)\,dr\,d\theta = 24\pi$$

$$\int_0^{2\pi} \frac{3r^4}{4}(1+3\sin 2\theta)\Big|_0^2 d\theta = \int_0^{2\pi} 12(1+3\sin 2\theta)\,d\theta$$

---

16. Use Green's theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = \left\langle \arctan x + y^2, e^{y} - x^2\right\rangle$ and C is the unit circle $x^2+y^2=1$, oriented clockwise.

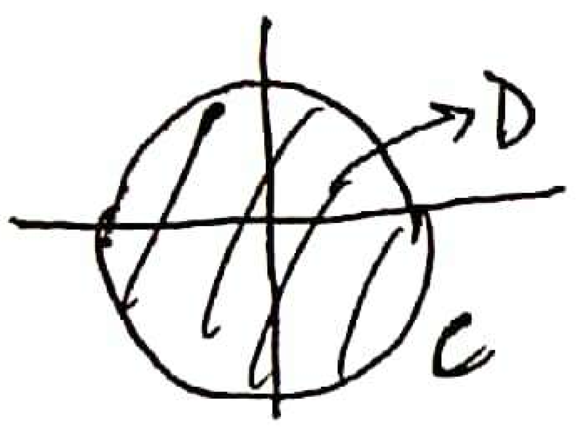
C: $x=\cos t$, $y=\sin t$, $0\leqslant t\leqslant 2\pi$

$M = \arctan x + y^2$

$N = e^y - x^2$

$M_y = 2y$

$N_x = -2x$



$$\int_C F\cdot dr = -\iint_D (-2x-2y)\,dA = 2\iint_D (x+y)\,r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = 2\int_0^{2\pi}(\cos\theta+\sin\theta)\frac{r^3}{3}\Big|_0^1 d\theta$$

$$= \frac{2}{3}(\sin\theta - \cos\theta)\Big|_0^{2\pi} = \frac{2}{3}\left[(0-1)-(0-1)\right] = 0,$$

---

17. Find the work done by the force field $\vec{F}(x,y,z) = 3x^2 i + (2xz - y)\,j + zk$ to move a particle along the straight line from $(0,0,0)$ to $(2,1,3)$.

We need to parametrize C

$r(t) = \langle 0,0,0\rangle + t\langle 2,1,3\rangle = \langle 2t, t, 3t\rangle$

$x = 2t$, $y = t$, $z = 3t$

$r'(t) = \langle 2,1,3\rangle$

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\vec{r}'(t)\,dt = \int_0^1 \langle 3(2t)^2, 2\cdot 2t\cdot 3t - t, 3t\rangle\cdot\langle 2,1,3\rangle\,dt$$

$$(24t^2 + 12t^2 - t + 9t)\,dt$$

$$\int_0^1 (36t^2+8t)\,dt = (12t^3+4t^2)\Big|_0^1 = 16.$$

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18. Find the work done by the force field $\vec{F}(x,y,z) = xi + yj + 3k$ on a particle that moves along the helix $r(t) = \langle \cos t, \sin t, t\rangle$ from $t=0$ to $t=2\pi$.

$x = \cos t$

$y = \sin t$

$z = t$

$r'(t) = \langle -\sin t, \cos t, 1\rangle$

$$\int_0^{2\pi} \langle \cos t, \sin t, 3\rangle\cdot\langle -\sin t, \cos t, 1\rangle\,dt$$

$$= \int_0^{2\pi} (-\cos t\sin t + \sin t\cos t + 3)\,dt$$

$$= \int_0^{2\pi} 3\,dt = 3t\Big|_0^{2\pi} = 6\pi.$$