Vectors

Definition

A vector is a quantity that has both magnitude and direction.

Definition

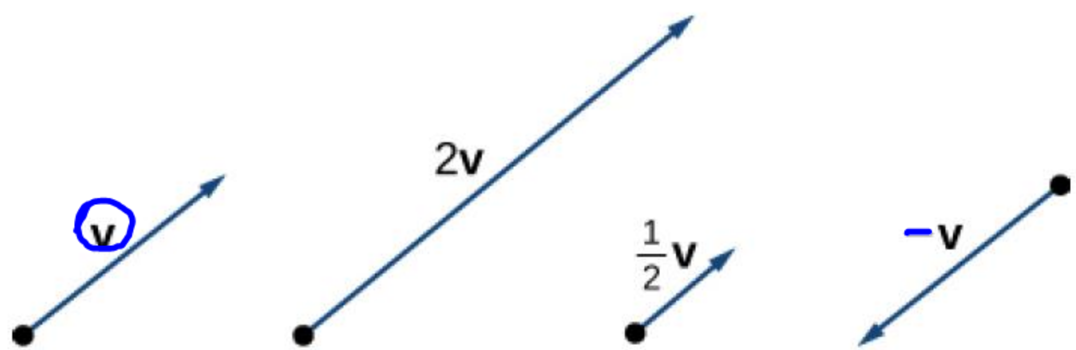
Vectors are said to be **equivalent vectors** if they have the same magnitude and direction.

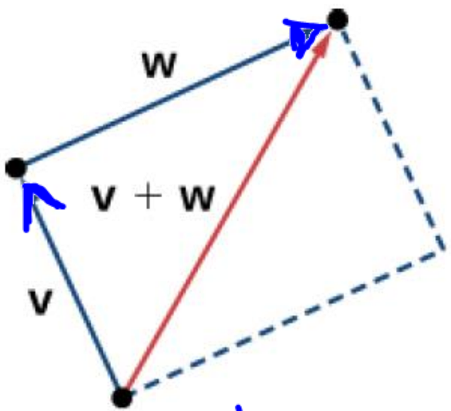
Definition

The product $k\mathbf{v}$ of a vector $\mathbf{v}$ and a scalar k is a vector with a magnitude that is $|k|$ times the magnitude of $\mathbf{v}$, and with a direction that is the same as the direction of $\mathbf{v}$ if $k > 0$, and opposite the direction of $\mathbf{v}$ if $k < 0$. This is called **scalar multiplication**. If $k = 0$ or $\mathbf{v} = \mathbf{0}$, then $k\mathbf{v} = \mathbf{0}$.

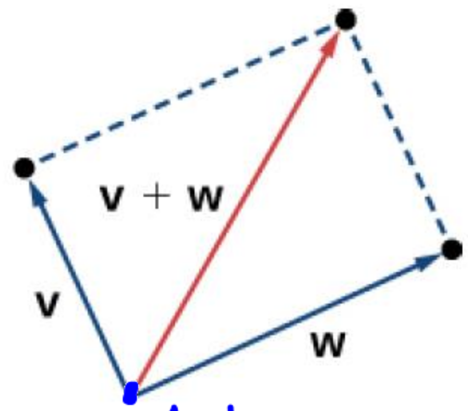
$\vec{v}$

bold v





triangle



parallelogram

$v = \langle 2, 3 \rangle$

$w = \langle 1, 4 \rangle$

$v + w = \langle 2+1, 3+4 \rangle$

triangle inequality.

$$\| \mathbf{v} + \mathbf{w} \| \leq \| \mathbf{v} \| + \| \mathbf{w} \|$$

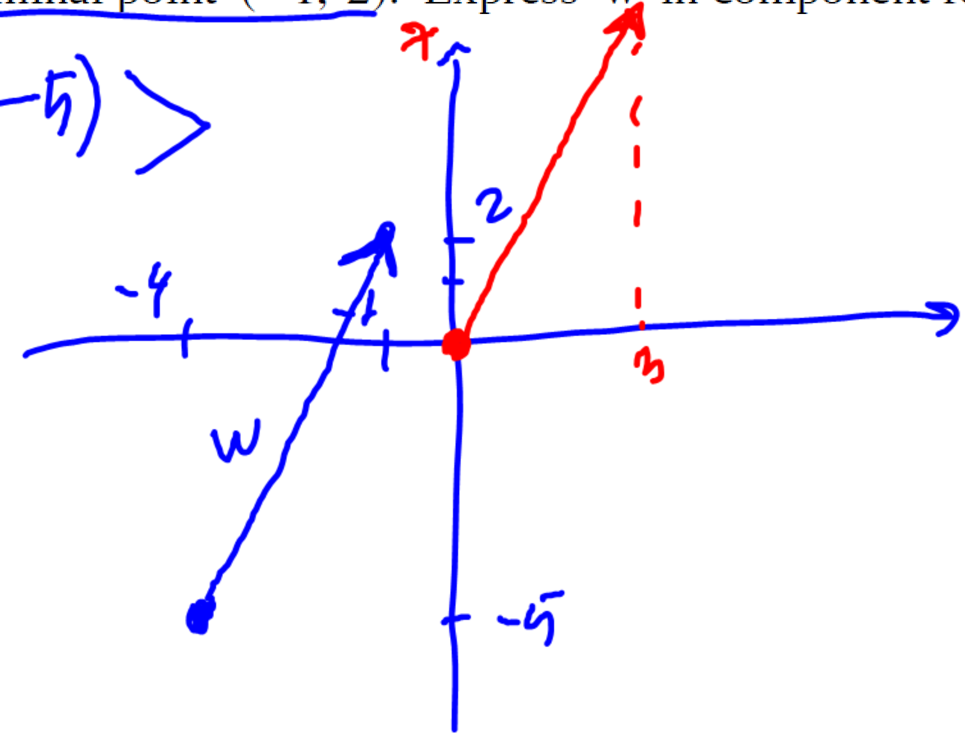
Rule: Component Form of a Vector

Let **v** be a vector with initial point (x_i, y_i) and terminal point (x_t, y_t). Then we can express **v** in component form as $\mathbf{v} = \langle x_t - x_i, y_t - y_i \rangle$.

2.4 Vector **w** has initial point $(-4, -5)$ and terminal point $(-1, 2)$. Express **w** in component form.

$$w = \langle -1-(-4), 2-(-5) \rangle$$

$$= \langle 3, 7 \rangle$$



Definition

Let $\mathbf{v} = \langle x_1, y_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2 \rangle$ be vectors, and let k be a scalar.

Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1 \rangle$ $\quad 2\langle 1,3\rangle = \langle 2,6\rangle$

Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + x_2, y_1 + y_2 \rangle$

2.5 Let $\mathbf{a} = \langle 7, 1 \rangle$ and let $\mathbf{b}$ be the vector with initial point $(3, 2)$ and terminal point $(-1, -1)$.

a. Find $\|\mathbf{a}\| = \sqrt{7^2+1^2} = 5\sqrt{2}$

b. Express $\mathbf{b}$ in component form.

$b = \langle -1-3, -1-2 \rangle$

$= \langle -4, -3 \rangle$

c. Find $3\mathbf{a} - 4\mathbf{b}$.

$3a - 4b = 3\langle 7,1\rangle - 4\langle -4,-3\rangle$

$= \langle 21, 3\rangle - \langle -16, -12\rangle = \langle 21+16, 3+12\rangle$

$= \langle 37, 15\rangle$

Theorem 2.1: Properties of Vector Operations

Let **u**, **v**, and **w** be vectors in a plane. Let r and s be scalars.

i.	$\mathbf{u}+\mathbf{v}$	$=$	$\mathbf{v}+\mathbf{u}$	Commutative property
ii.	$(\mathbf{u}+\mathbf{v})+\mathbf{w}$	$=$	$\mathbf{u}+(\mathbf{v}+\mathbf{w})$	Associative property
iii.	$\mathbf{u}+0$	$=$	$\mathbf{u}$	Additive identity property
iv.	$\mathbf{u}+(-\mathbf{u})$	$=$	0	Additive inverse property
v.	$r(s\mathbf{u})$	$=$	$(rs)\mathbf{u}$	Associativity of scalar multiplication
vi.	$(r+s)\mathbf{u}$	$=$	$r\mathbf{u}+s\mathbf{u}$	Distributive property
vii.	$r(\mathbf{u}+\mathbf{v})$	$=$	$r\mathbf{u}+r\mathbf{v}$	Distributive property
viii.	$1\mathbf{u}$	$=$	$\mathbf{u},\ 0\mathbf{u}=\vec{0}$	Identity and zero properties

$\vec{0} = \langle 0, 0 \rangle$

A **unit vector** is a vector with magnitude 1.

$$\frac{\vec{v}}{\|\vec{v}\|}$$

$\mathbf{u} = \dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$ We say that $\mathbf{u}$ is the *unit vector in the direction of* $\mathbf{v}$

2.8 Let $\mathbf{v} = \langle 9, 2 \rangle$. Find a vector with magnitude 5 in the opposite direction as $\mathbf{v}$.

$$\frac{v}{\|v\|} = \frac{\langle 9, 2\rangle}{\sqrt{9^2+2^2}} = \frac{\langle 9, 2\rangle}{\sqrt{85}}$$

$$w = 5\left(-\frac{v}{\|v\|}\right) = -\frac{5}{\sqrt{85}}\langle 9, 2\rangle = \left\langle -\frac{9\sqrt{85}}{17}, -\frac{2\sqrt{85}}{17} \right\rangle$$

unit vector
in the opposite
direction of v

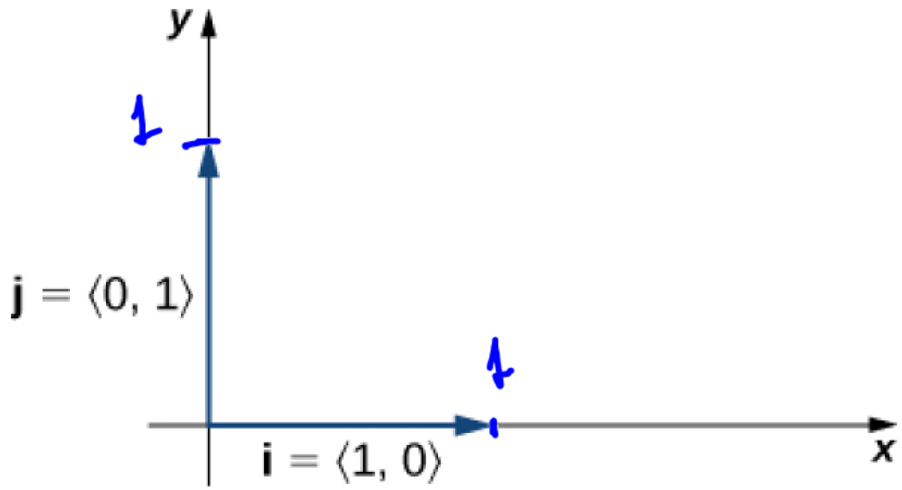
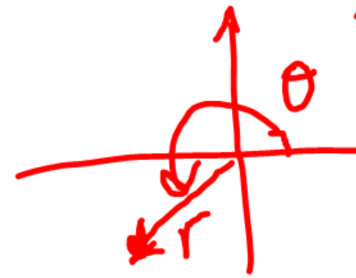


Figure 2.18 The standard unit vectors **i** and **j**.

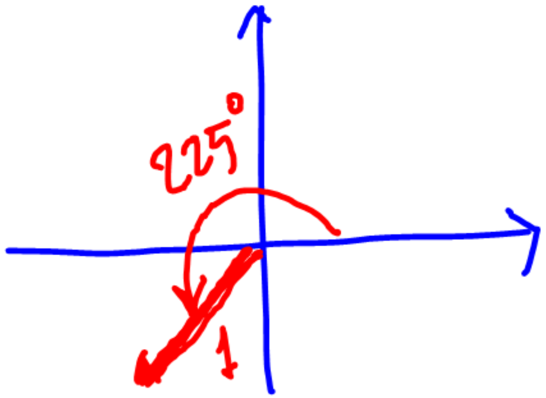
$$\mathbf{v} = \langle x, y \rangle = \langle x, 0 \rangle + \langle 0, y \rangle = x\langle 1, 0 \rangle + y\langle 0, 1 \rangle = x\mathbf{i} + y\mathbf{j}.$$

2.9 Let $\mathbf{a} = \langle\, 16, -11 \,\rangle$ and let $\mathbf{b}$ be a unit vector that forms an angle of $225°$ with the positive x-axis. Express $\mathbf{a}$ and $\mathbf{b}$ in terms of the standard unit vectors.

$$\langle r\cos\theta, r\sin\theta \rangle$$



$$a = 16\mathbf{i} - 11\mathbf{j}$$



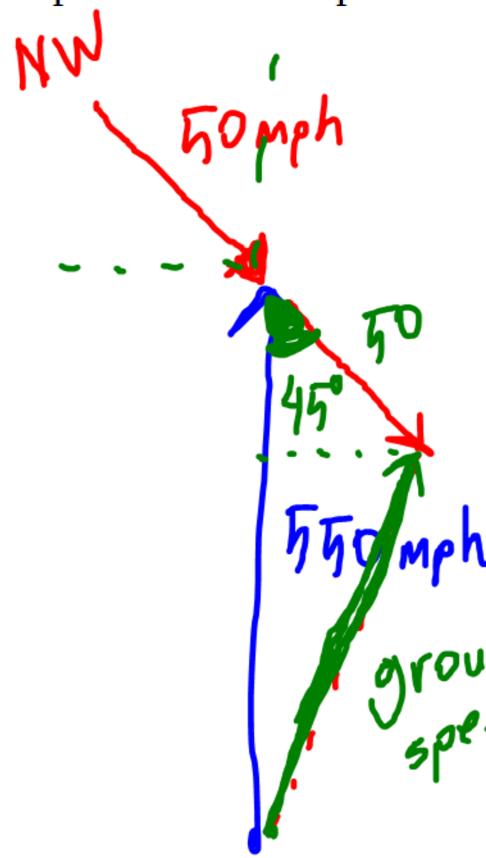
$\langle \cos\theta, \sin\theta \rangle$ is the unit vector in the direction of θ.

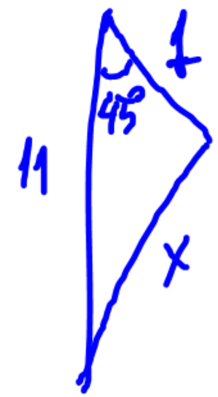
$$b = \langle \cos 225°, \sin 225° \rangle$$

$$= \langle -\cos 45°, -\sin 45° \rangle = \left\langle -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right\rangle$$

2.10 An airplane flies due north at an airspeed of 550 mph. The wind is blowing from the northwest at 50 mph. What is the ground speed of the airplane?

$$a^2 = b^2 + c^2 - 2bc\cos A$$





cosine rule

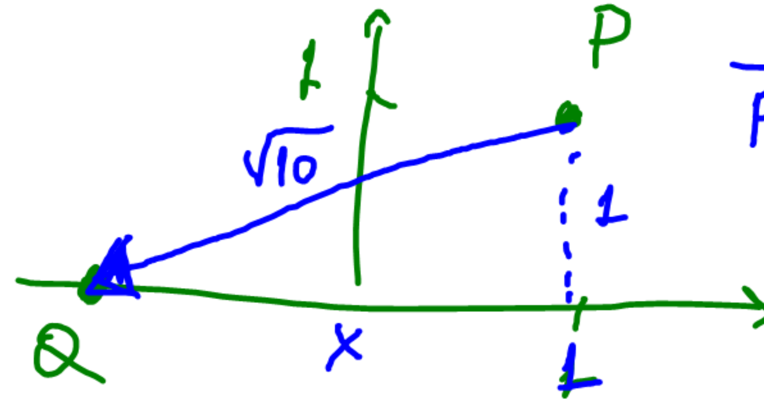
$$x^2 = 11^2 + 1^2 - 2 \times 11 \times 1 \times \cos 45^\circ$$

$$x^2 = 122 - 11\sqrt{2}$$

$$x = \sqrt{122 - 11\sqrt{2}}$$

ground speed is $50x = 50\sqrt{122 - 11\sqrt{2}}$

14. The vector **v** has initial point $P(1,\ 1)$ and terminal point Q that is on the x-axis and left of the initial point. Find the coordinates of terminal point Q such that the magnitude of the vector **v** is $\sqrt{10}$.



$\overrightarrow{PQ}$

$$\sqrt{10}^{2} = 1^2 + x^2$$

$$x = 3$$

$Q(-2, 0)$

43. Calculate the coordinates of point D such that $ABCD$ is a parallelogram, with $A(1, 1)$, $B(2, 4)$, and $C(7, 4)$.

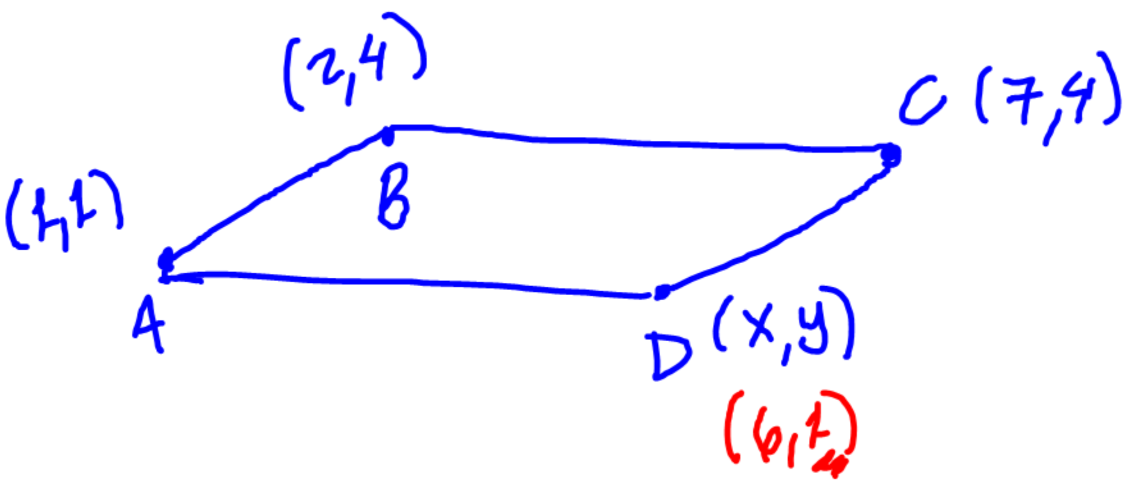
$\vec{AB}$ is equivalent to $\vec{DC}$

$\vec{AD}$ " $\vec{BC}$

$\vec{AD} = \langle x-1, y-1 \rangle$

$\vec{BC} = \vec{C} - \vec{B} = \langle 7-2, 4-4 \rangle$

$= \langle 5, 0 \rangle$



$\vec{AD} = \langle x-1, y-1 \rangle = \langle 5, 0 \rangle$

$x-1=5$

$y-1=0$

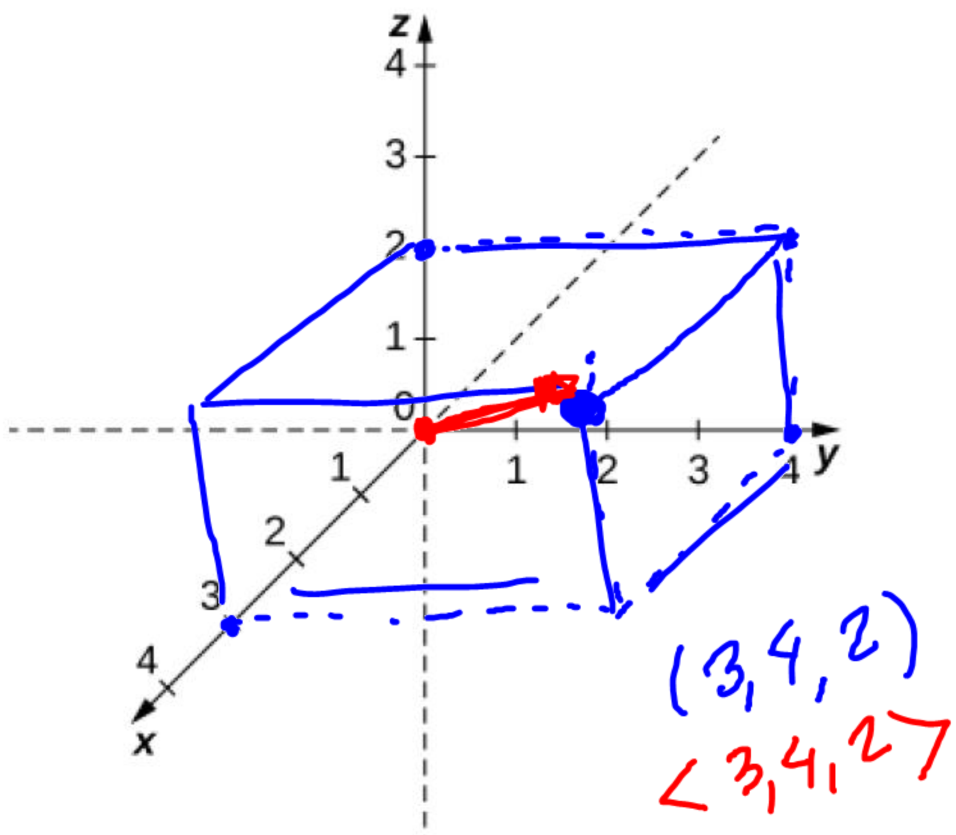
$x=6$

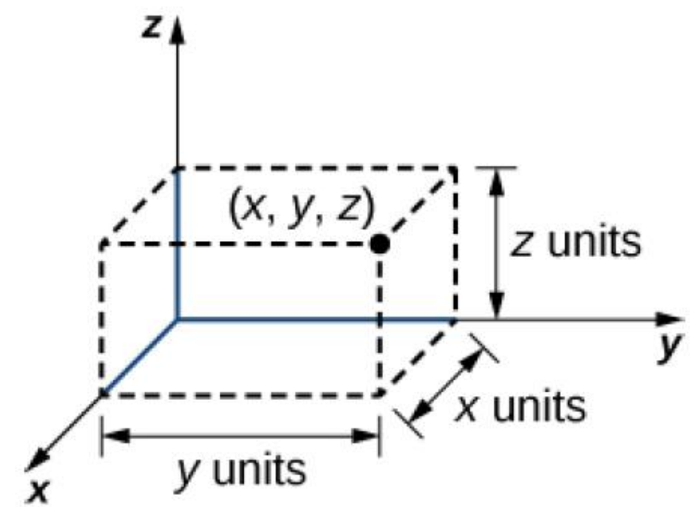
$y=1$

Definition

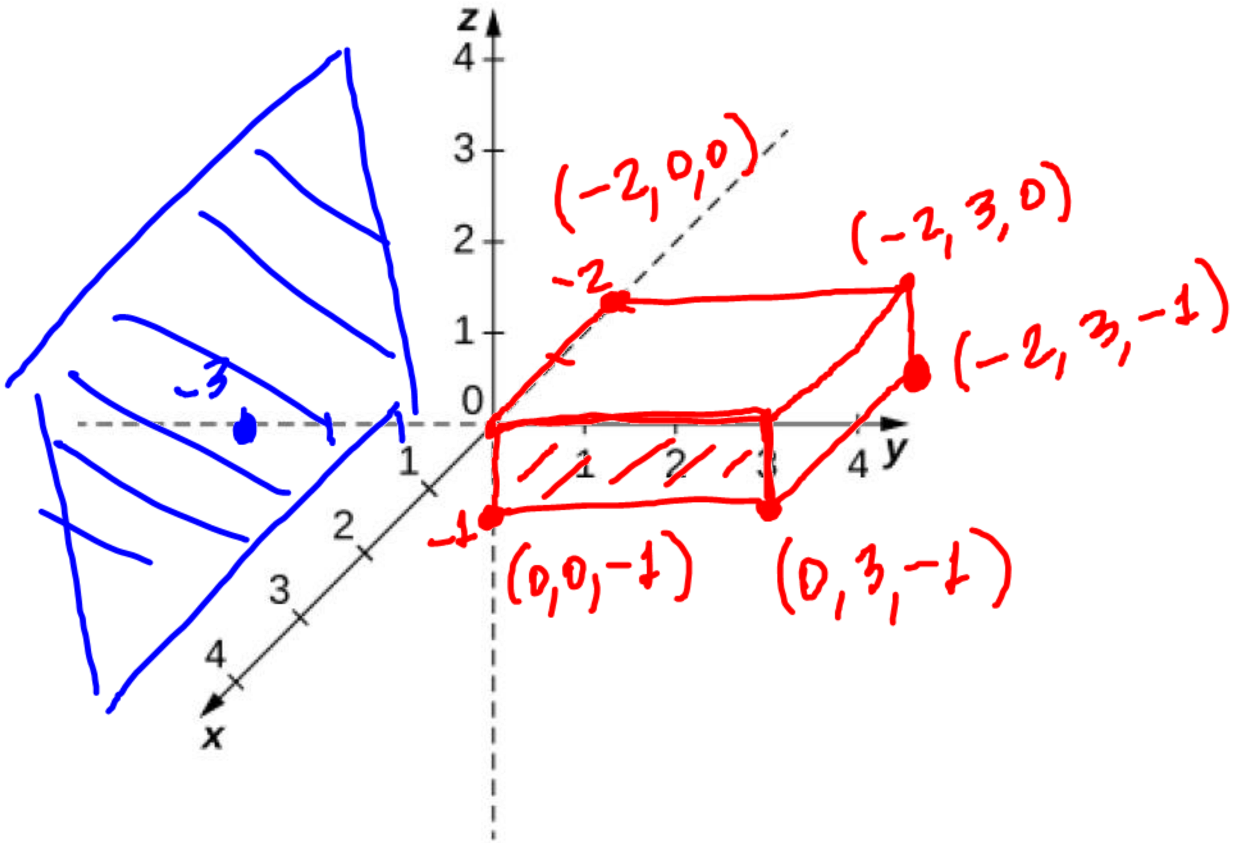
The **three-dimensional rectangular coordinate system** consists of three perpendicular axes: the *x*-axis, the *y*-axis, and the *z*-axis. Because each axis is a number line representing all real numbers in $\mathbb{R}$, the three-dimensional system is often denoted by $\mathbb{R}^3$. 3-space

$\mathbb{R}^2$ plane





2.11 Sketch the point $(-2, 3, -1)$ in three-dimensional space.



$y=-3$ is a point in 2-dimension

$y=-3$ is a plane in 3-dimension

(x and z are free)

$(x, -3, z)$

parallel to xz plane.

Theorem 2.2: The Distance between Two Points in Space

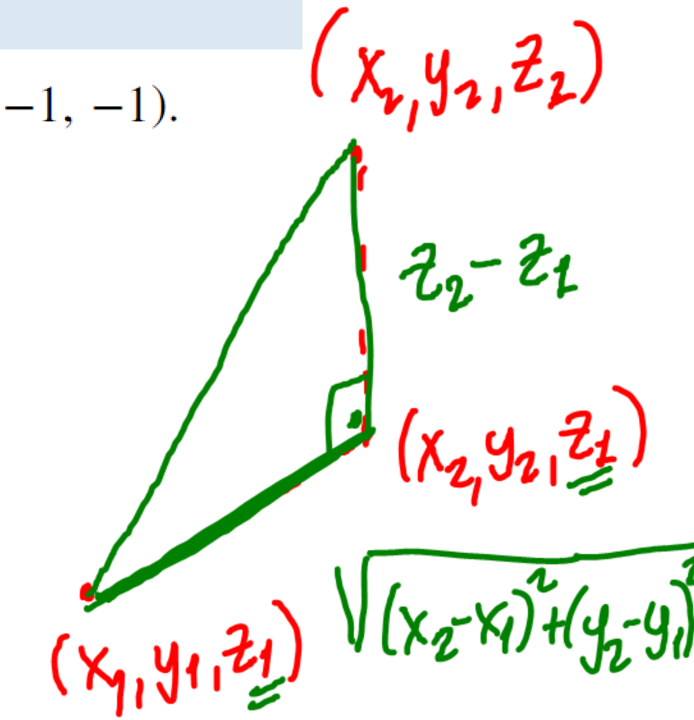
The distance d between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

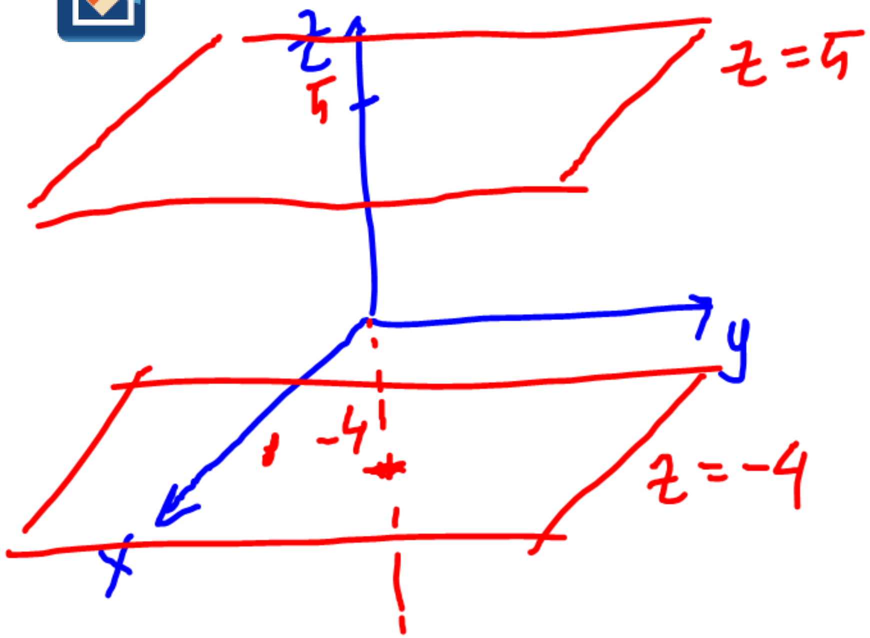
2.12 Find the distance between points $P_1 = (1, -5, 4)$ and $P_2 = (4, -1, -1)$.

$$d = \sqrt{(4-1)^2 + (-1-(-5))^2 + (-1-4)^2}$$

$$= \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$$



2.13 Write an equation of the plane passing through point $(1, -6, -4)$ that is parallel to the xy-plane.



$z = a$ is a plane parallel to xy-plane.

$z = -4$ is the plane parallel to xy-plane

it passes through $(1, -6, -4)$

Rule: Equation of a Sphere

The sphere with center (a, b, c) and radius r can be represented by the equation

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2.$$

This equation is known as the **standard equation of a sphere.**

Rule: Equation of a Sphere

The sphere with center (a, b, c) and radius r can be represented by the equation

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

This equation is known as the **standard equation of a sphere.**

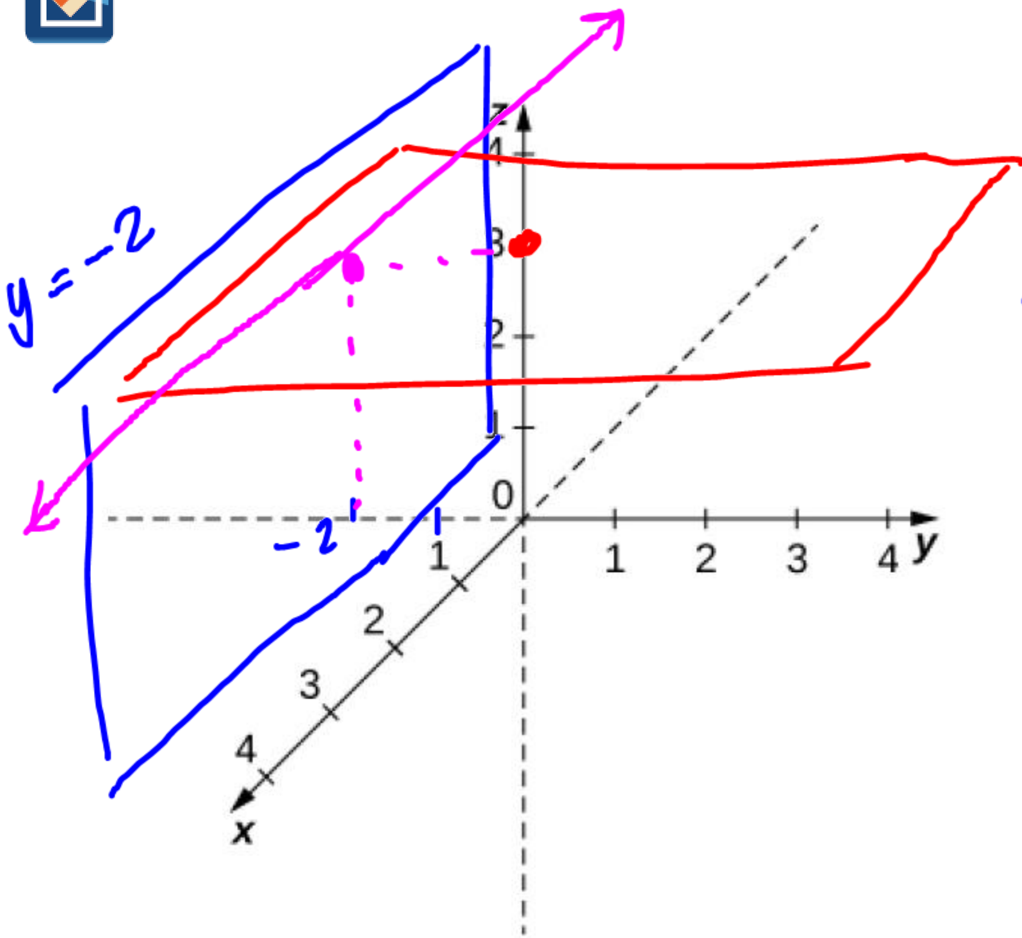
2.14 Find the standard equation of the sphere with center $(-2, 4, -5)$ containing point $(4, 4, -1)$.

C P

$$r = |CP| = \sqrt{(4-(-2))^2 + (4-4)^2 + (-1-(-5))^2} = \sqrt{36+0+16} = \sqrt{52}$$

$$(x+2)^2 + (y-4)^2 + (z+5)^2 = r^2 = 52$$

2.16 Describe the set of points that satisfies $(y+2)(z-3)=0$, and graph the set.



$y+2=0$ OR $z-3=0$

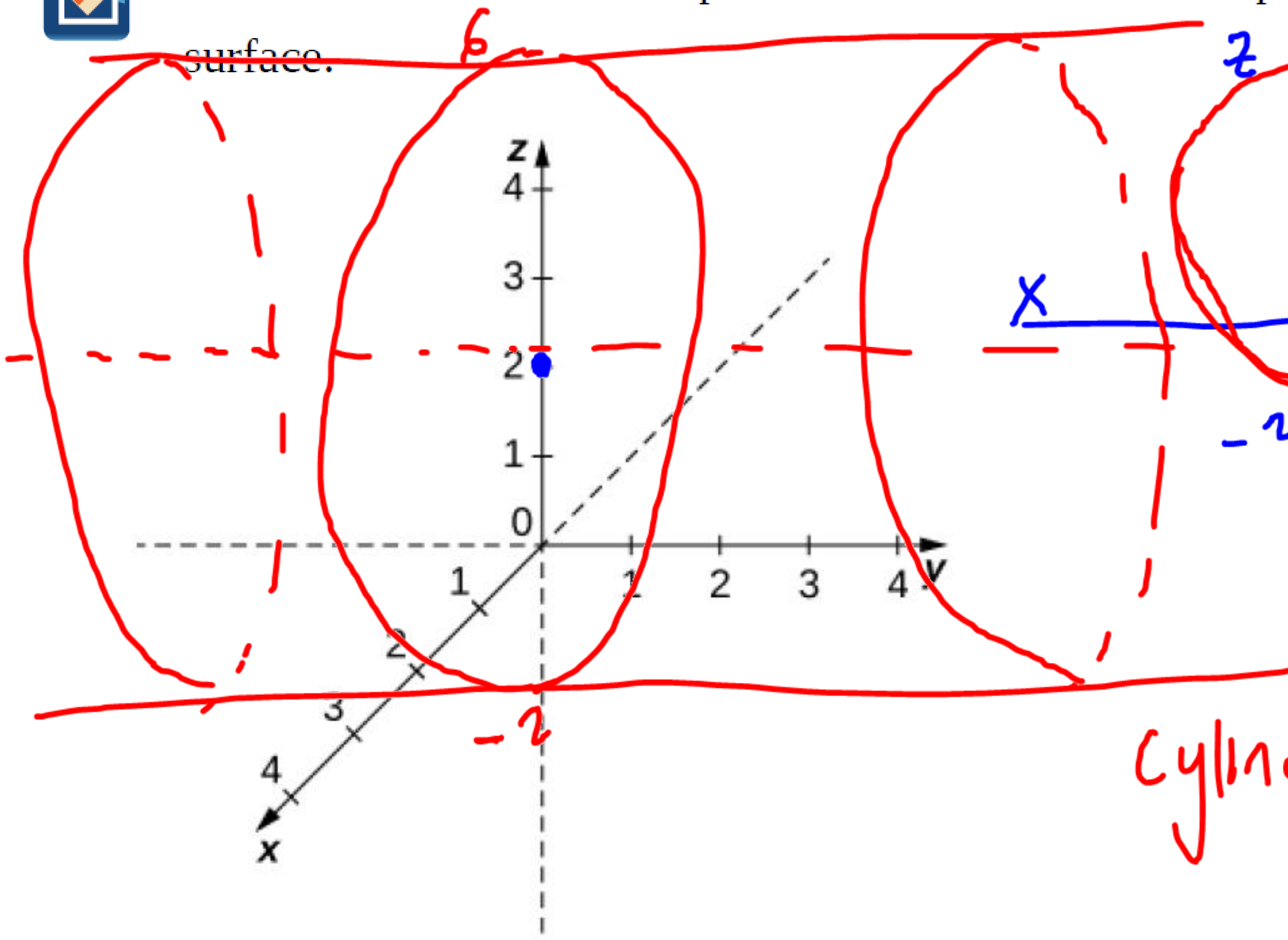
$y=-2$ OR $z=3$

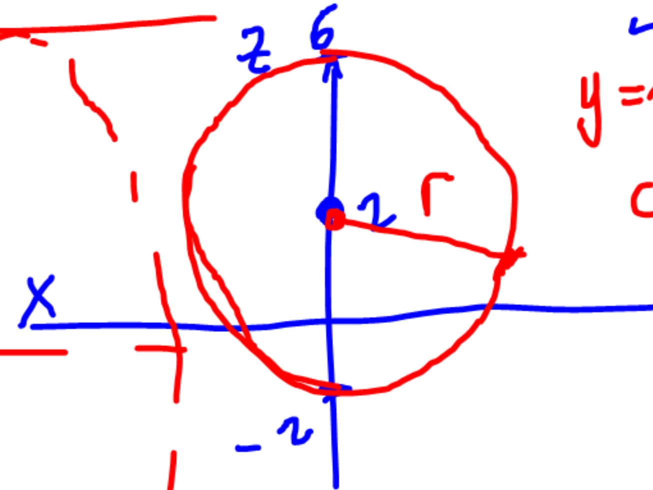
plane parallel to xz

$z=3$ is a plane parallel to xy.

$(x,-2,3)$ is the intersection which is a line

2.17 Describe the set of points in three dimensional space that satisfies $x^2 + (z-2)^2 = 16$, and graph the surface.





y=0 then it is a circle on xz-plane.

cylinder

2.18 Let $S = (3, 8, 2)$ and $T = (2, -1, 3)$. Express $\overrightarrow{ST}$ in component form and in standard unit form.

$$\overrightarrow{ST} = \vec{T} - \vec{S} = \langle 2-3, -1-8, 3-2 \rangle$$

$$= \langle -1, -9, 1 \rangle$$

$$= -i - 9j + k$$

$i = \langle 1, 0, 0 \rangle$

$j = \langle 0, 1, 0 \rangle$

$k = \langle 0, 0, 1 \rangle$

Rule: Properties of Vectors in Space

Let $\mathbf{v} = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2, z_2 \rangle$ be vectors, and let k be a scalar.

Scalar multiplication: $k\mathbf{v} = \langle kx_1, ky_1, kz_1 \rangle$

Vector addition: $\mathbf{v} + \mathbf{w} = \langle x_1, y_1, z_1 \rangle + \langle x_2, y_2, z_2 \rangle = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$

Vector subtraction: $\mathbf{v} - \mathbf{w} = \langle x_1, y_1, z_1 \rangle - \langle x_2, y_2, z_2 \rangle = \langle x_1 - x_2, y_1 - y_2, z_1 - z_2 \rangle$

Vector magnitude: $\|\mathbf{v}\| = \sqrt{{x_1}^2 + {y_1}^2 + {z_1}^2}$

Unit vector in the direction of v: $\dfrac{1}{\|\mathbf{v}\|}\mathbf{v} = \dfrac{1}{\|\mathbf{v}\|}\langle x_1, y_1, z_1 \rangle = \left\langle \dfrac{x_1}{\|\mathbf{v}\|}, \dfrac{y_1}{\|\mathbf{v}\|}, \dfrac{z_1}{\|\mathbf{v}\|} \right\rangle,$ if $\mathbf{v} \neq \mathbf{0}$

2.19 Let $\mathbf{v} = \langle -1, -1, 1 \rangle$ and $\mathbf{w} = \langle 2, 0, 1 \rangle$. Find a unit vector in the direction of $5\mathbf{v} + 3\mathbf{w}$.

$$5v + 3w = 5\langle -1, -1, 1\rangle + 3\langle 2, 0, 1\rangle$$

$$= \langle -5, -5, 5\rangle + \langle 6, 0, 3\rangle = \langle 1, -5, 8\rangle$$

unit vector in that direction. $= \dfrac{\langle 1, -5, 8\rangle}{\sqrt{1^2+5^2+8^2}} = \dfrac{\langle 1, -5, 8\rangle}{\sqrt{1+25+64}} = \dfrac{\langle 1, -5, 8\rangle}{3\sqrt{10}}$

$$= \frac{\sqrt{10}}{30}\langle 1, -5, 8\rangle = \left\langle \frac{\sqrt{10}}{30}, -\frac{\sqrt{10}}{6}, \frac{4\sqrt{10}}{15} \right\rangle$$

80. $Q(0, 7, -6)$ and $M(-1, 3, 2)$, where M is the midpoint of the line segment PQ

$P = ?$

$P(x, y, z)$

$-1 = \frac{0+x}{2}$ $\quad 3 = \frac{7+y}{2}$ $\quad 2 = \frac{-6+z}{2}$

$x = -2$ $\quad y = -1$ $\quad z = 10$

$P(-2, -1, 10)$

82. Find initial point P of vector $\vec{PQ} = \langle -9, 1, 2 \rangle$ with the terminal point at $Q(10, 0, -1)$.

$\vec{PQ} = \vec{Q} - \vec{P} = \langle 10, 0, -1 \rangle - \langle x, y, z \rangle = \langle -9, 1, 2 \rangle$

$\langle 10 - x, -y, -1 - z \rangle = \langle -9, 1, 2 \rangle$

$x = 19, \quad y = -1, \quad z = -3$

109. The points A, B, and C are collinear (in this order) if the relation $\| \overrightarrow{AB} \| + \| \overrightarrow{BC} \| = \| \overrightarrow{AC} \|$ is satisfied. Show that $A(5, 3, -1)$, $B(-5, -3, 1)$, and $C(-15, -9, 3)$ are collinear points.

110. Show that points $A(1, 0, 1)$, $B(0, 1, 1)$, and $C(1, 1, 1)$ are not collinear.