Module 2
vector valued functions

Chapter 3: Vector-Valued Functions . . . .
3.1 Vector-Valued Functions and Space Curves .
3.2 Calculus of Vector-Valued Functions . . . . .
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Definition

A **vector-valued function** is a function of the form

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \quad \text{or} \quad \mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, \tag{3.1}$$

where the **component functions** f, g, and h, are real-valued functions of the parameter t. Vector-valued functions are also written in the form

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle \quad \text{or} \quad \mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle . \tag{3.2}$$

In both cases, the first form of the function defines a two-dimensional vector-valued function; the second form describes a three-dimensional vector-valued function.

$r(t) = \langle \sin t, \cos t \rangle$

$x(t) = \sin t$

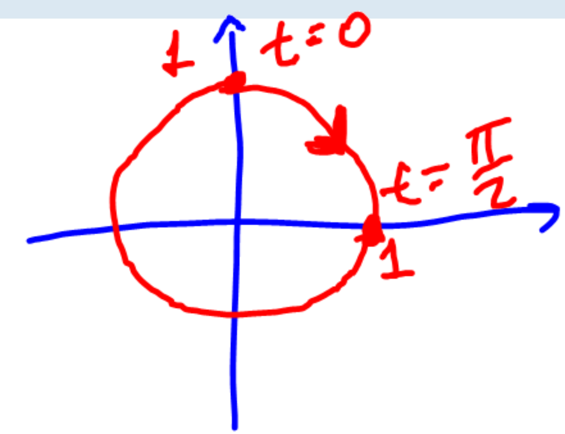
$y(t) = \cos t$

$x^2 + y^2 = \sin^2 t + \cos^2 t = 1$

circle

$r(0) = \langle 0, 1 \rangle$

$r(\frac{\pi}{2}) = \langle 1, 0 \rangle$



3.1 For the vector-valued function $\mathbf{r}(t) = \left(t^2 - 3t\right)\mathbf{i} + (4t + 1)\,\mathbf{j}$, evaluate $\mathbf{r}(0)$, $\mathbf{r}(1)$, and $\mathbf{r}(-4)$. Does this function have any domain restrictions? NO

$r(0) = \langle 0, 1 \rangle$

$r(1) = \langle -2, 5 \rangle$

$r(-4) = \langle 28, -15 \rangle$

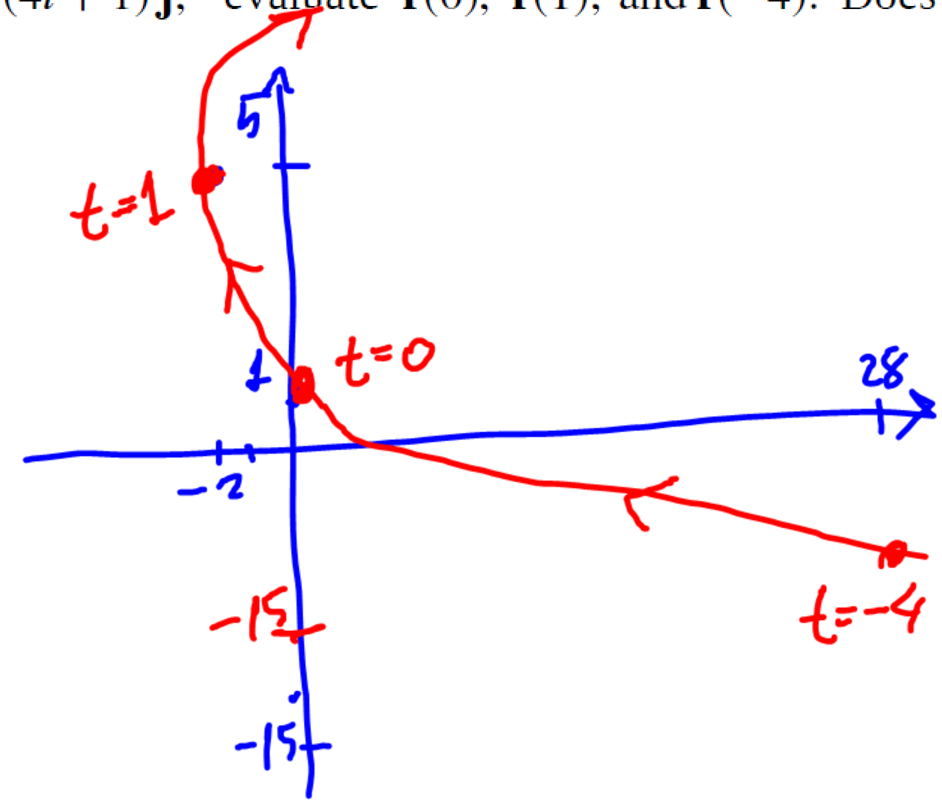
$y = 4t + 1$

$\frac{y-1}{4} = t$

$x(t) = t^2 - 3t$

$x = \left(\frac{y-1}{4}\right)^2 - 3\left(\frac{y-1}{4}\right)$

parabola



3.2 Create a graph of the vector-valued function $\mathbf{r}(t) = \left(t^2 - 1\right)\mathbf{i} + (2t - 3)\,\mathbf{j}, \quad 0 \le t \le 3.$

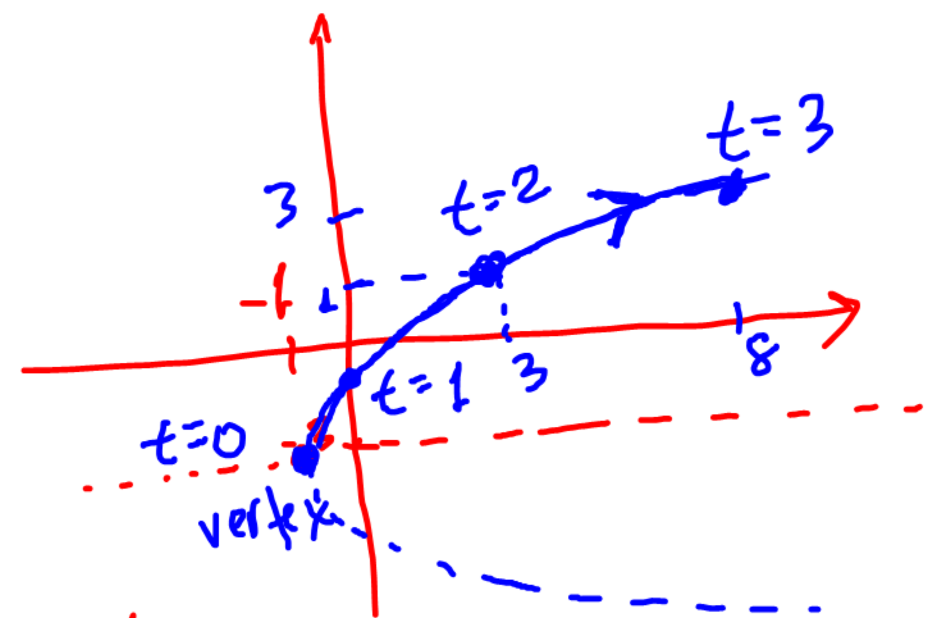
$y = 2t - 3 \qquad t = \frac{y+3}{2}$

$x = t^2 - 1 = \left(\frac{y+3}{2}\right)^2 - 1$

$x = \frac{y^2 + 6y + 5}{4}$

parabola

vertex $-\frac{b}{2a} = -\frac{6}{2} = -3$



$y = -3$ is the axis of symmetry

$r(0) = \langle -1, -3 \rangle$

$r(1) = \langle 0, -1 \rangle$

$r(3) = \langle 8, 3 \rangle$

$r(2) = \langle 3, 1 \rangle$

Definition

A vector-valued function $\mathbf{r}$ approaches the limit $\mathbf{L}$ as t approaches a, written

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L},$$

provided

$$\lim_{t \to a} \|\mathbf{r}(t) - \mathbf{L}\| = 0.$$

Theorem 3.1: Limit of a Vector-Valued Function

Let f, g, and h be functions of t. Then the limit of the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ as t approaches a is given by

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j}, \tag{3.3}$$

provided the limits $\lim_{t \to a} f(t)$ and $\lim_{t \to a} g(t)$ exist. Similarly, the limit of the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ as t approaches a is given by

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j} + \left[\lim_{t \to a} h(t)\right]\mathbf{k}, \tag{3.4}$$

provided the limits $\lim_{t \to a} f(t)$, $\lim_{t \to a} g(t)$and $\lim_{t \to a} h(t)$ exist.

11. $\lim_{t \to e^2} \langle\, t \ln(t), \frac{\ln t}{t^2}, \sqrt{\ln\left(t^2\right)}\,\rangle$ $= \left\langle e^2 \ln e^2, \frac{\ln e^2}{(e^2)^2}, \sqrt{\ln (e^2)^2} \right\rangle$

$\langle 2e^2, 2/e^4, 2 \rangle$

12. $\lim_{t \to \pi/6} \langle\, \cos^2 t, \sin^2 t, 1\,\rangle$ $= \left\langle \left(\frac{\sqrt{3}}{2}\right)^2, \left(\frac{1}{2}\right)^2, 1 \right\rangle$

$= \left\langle \frac{3}{4}, \frac{1}{4}, 1 \right\rangle$

3.3 Calculate $\lim_{t \to -2} \mathbf{r}(t)$ for the function $\mathbf{r}(t) = \sqrt{t^2 - 3t - 1}\,\mathbf{i} + (4t+3)\,\mathbf{j} + \sin\frac{(t+1)\pi}{2}\mathbf{k}$.

$$\left\langle \sqrt{4+6-1},\ 4(-2)+3,\ \sin\left(-\frac{\pi}{2}\right) \right\rangle$$

$$= \langle 3, -5, -1 \rangle$$

Definition

Let f, g, and h be functions of t. Then, the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ is continuous at point $t = a$ if the following three conditions hold:

1. $\mathbf{r}(a)$ exists
2. $\lim\limits_{t \to a} \mathbf{r}(t)$ exists
3. $\lim\limits_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$

Similarly, the vector-valued function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is continuous at point $t = a$ if the following three conditions hold:

1. $\mathbf{r}(a)$ exists
2. $\lim\limits_{t \to a} \mathbf{r}(t)$ exists
3. $\lim\limits_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$

6. Given the vector-valued function $\mathbf{r}(t) = \langle\, t,\ t^2 + 1\, \rangle$, find the following values:

a. $\lim_{t \to -3} \mathbf{r}(t)$ $= \langle -3, 10 \rangle$

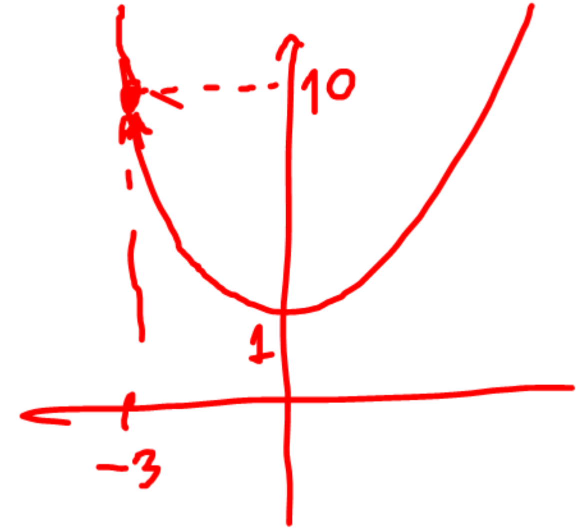
b. $\mathbf{r}(-3)$ $= \langle -3, 10 \rangle$

c. Is $\mathbf{r}(t)$ continuous at $x = -3$? YES

d. $\mathbf{r}(t + 2) - \mathbf{r}(t)$

$t = x$

$y = x^2 + 1$



$\langle t+2, (t+2)^2+1 \rangle - \langle t, t^2+1 \rangle$

$= \langle 2, 4t+4 \rangle$

7. Let $\mathbf{r}(t) = e^t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln t\,\mathbf{k}$. Find the following values:

a. $\mathbf{r}\left(\frac{\pi}{4}\right)$

b. $\lim_{t \to \pi/4} \mathbf{r}(t)$

c. Is $\mathbf{r}(t)$ continuous at $t = t = \frac{\pi}{4}$?

$r\left(\frac{\pi}{4}\right) = e^{\pi/4}\,i + \frac{\sqrt{2}}{2}\,j + \ln\left(\frac{\pi}{4}\right)k$

$\lim_{t \to \frac{\pi}{4}} r(t) = r\left(\frac{\pi}{4}\right)$

YES.

$\mathrm{Dom}(r(t)) = (0, \infty)$

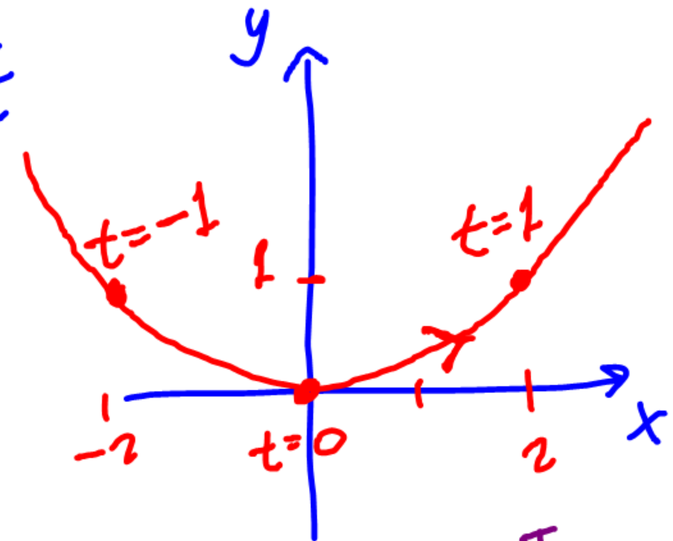
Eliminate the parameter t, write the equation in Cartesian coordinates, then sketch the graphs of the vector-valued functions.

22. $\mathbf{r}(t) = 2t\mathbf{i} + t^2\,\mathbf{j}$ (*Hint:* Let $x = 2t$ and $y = t^2$. Solve the first equation for x in terms of t and substitute this result into the second equation.)

$x = 2t \qquad \frac{x}{2} = t$

$y = t^2$

$y = \left(\frac{x}{2}\right)^2$



$r(t) = \langle 2\cos t, 3\sin t \rangle$

$x = 2\cos t \qquad \cos t = \frac{x}{2}$

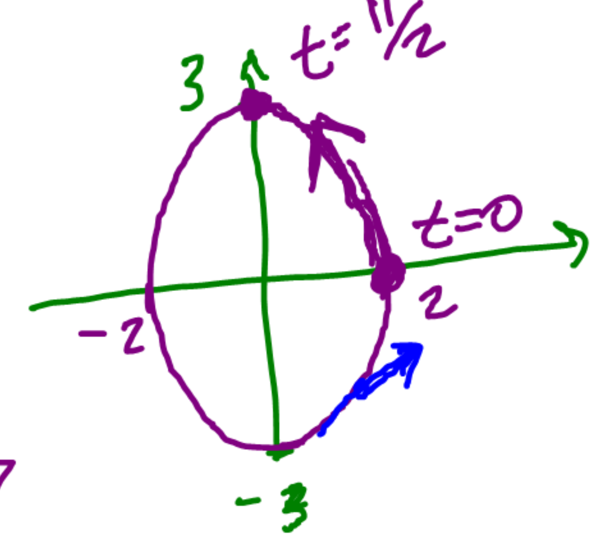
$y = 3\sin t \qquad \sin t = \frac{y}{3}$

$\cos^2 t + \sin^2 t = 1$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$r(0) = \langle 2, 0 \rangle$



Definition

The **derivative of a vector-valued function** $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}, \tag{3.5}$$

provided the limit exists. If $\mathbf{r}'(t)$ exists, then $\mathbf{r}$ is differentiable at t. If $\mathbf{r}'(t)$ exists for all t in an open interval (a, b), then $\mathbf{r}$ is differentiable over the interval (a, b). For the function to be differentiable over the closed interval $[a, b]$, the following two limits must exist as well:

$$\mathbf{r}'(a) = \lim_{\Delta t \to 0^+} \frac{\mathbf{r}(a + \Delta t) - \mathbf{r}(a)}{\Delta t} \text{ and } \mathbf{r}'(b) = \lim_{\Delta t \to 0^-} \frac{\mathbf{r}(b + \Delta t) - \mathbf{r}(b)}{\Delta t}.$$

3.4 Use the definition to calculate the derivative of the function $\mathbf{r}(t) = \left(2t^2 + 3\right)\mathbf{i} + (5t - 6)\,\mathbf{j}$.

$$r'(t) = \lim_{\Delta t \to 0} \frac{\langle 2(t+\Delta t)^2+3,\ 5(t+\Delta t)-6\rangle - \langle 2t^2+3,\ 5t-6\rangle}{\Delta t}$$

$$= \lim \frac{\langle \cancel{2t^2} + 4t\Delta t + 2\Delta t^2 + \cancel{3} - \cancel{2t^2} - \cancel{3},\ \cancel{5t} + 5\Delta t - \cancel{6} - \cancel{5t} + \cancel{6}\rangle}{\Delta t}$$

$$= \lim \frac{\langle 4t\Delta t + 2\Delta t^2,\ 5\Delta t\rangle}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta t\langle 4t + 2\Delta t,\ 5\rangle}{\Delta t}$$

$$= \langle 4t, 5\rangle = r'(t)$$

Theorem 3.2: Differentiation of Vector-Valued Functions

Let f, g, and h be differentiable functions of t.

i. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$.

ii. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$.

3.5 Calculate the derivative of the function

$$\mathbf{r}(t) = (t \ln t)\mathbf{i} + \left(5e^t\right)\mathbf{j} + (\cos t - \sin t)\mathbf{k}.$$

$$r'(t) = (1 \cdot \ln t + 1)\, i + 5e^t\, j + (-\sin t - \cos t)\, k$$

Theorem 3.3: Properties of the Derivative of Vector-Valued Functions

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let f be a differentiable real-valued function of t, and let c be a scalar.

i.	$\frac{d}{dt}[c\,\mathbf{r}(t)]$	$=$	$c\,\mathbf{r}'(t)$	Scalar multiple
ii.	$\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)]$	$=$	$\mathbf{r}'(t) \pm \mathbf{u}'(t)$	Sum and difference
iii.	$\frac{d}{dt}[f(t)\mathbf{u}(t)]$	$=$	$f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$	Scalar product f(t) is scalar f!
iv.	$\frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)]$	$=$	$\mathbf{r}'(t)\cdot\mathbf{u}(t) + \mathbf{r}(t)\cdot\mathbf{u}'(t)$	Dot product
v.	$\frac{d}{dt}[\mathbf{r}(t)\times\mathbf{u}(t)]$	$=$	$\mathbf{r}'(t)\times\mathbf{u}(t) + \mathbf{r}(t)\times\mathbf{u}'(t)$	Cross product
vi.	$\frac{d}{dt}[\mathbf{r}(f(t))]$	$=$	$\mathbf{r}'(f(t))\cdot f'(t)$	Chain rule
vii.	If $\mathbf{r}(t)\cdot\mathbf{r}(t)$	$=$	c, then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$.	

3.6 Given the vector-valued functions $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} - e^{2t}\,\mathbf{k}$ and $\mathbf{u}(t) = t\,\mathbf{i} + \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$, calculate $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}'(t)]$ and $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{r}(t)]$.

$r(t) = \langle \cos t, \sin t, -e^{2t} \rangle$

$r'(t) = \langle -\sin t, \cos t, -2e^{2t} \rangle$

$r(t) \cdot r'(t) = -\cos t \sin t + \sin t \cos t + 2e^{4t} = 2e^{4t}$

scalar function

$$\frac{d}{dt}[r(t) \cdot r'(t)] = 8e^{4t}.$$

$$u(t)\times r(t)=\begin{vmatrix} i & j & k \\ t & \sin t & \cos t \\ \cos t & \sin t & -e^{2t} \end{vmatrix} = i\sin t\left(-e^{2t}-\cos t\right)+j\left(te^{2t}+\cos^2 t\right)$$

$$+k\sin t\left(t-\cos t\right)$$

$$\frac{d}{dt}\left(u(t)\times r(t)\right)=\Big\langle \cos t\left(-e^{2t}-\cos t\right)+\sin t\left(-2e^{2t}+\sin t\right),$$

$$1.e^{2t}+t.2e^{2t}+2\cos t\left(-\sin t\right),$$

$$\cos t\left(t-\cos t\right)+\sin t\left(1+\sin t\right)\Big\rangle$$

Definition

Let C be a curve defined by a vector-valued function **r**, and assume that $\mathbf{r}'(t)$ exists when $t = t_0$. A tangent vector **v** at $t = t_0$ is any vector such that, when the tail of the vector is placed at point $\mathbf{r}(t_0)$ on the graph, vector **v** is tangent to curve C. Vector $\mathbf{r}'(t_0)$ is an example of a tangent vector at point $t = t_0$. Furthermore, assume that $\mathbf{r}'(t) \neq \mathbf{0}$. The **principal unit tangent vector** at t is defined to be

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \tag{3.6}$$

provided $\|\mathbf{r}'(t)\| \neq 0$.

3.7 Find the unit tangent vector for the vector-valued function

$$\mathbf{r}(t) = \left(t^2 - 3\right)\mathbf{i} + (2t + 1)\,\mathbf{j} + (t - 2)\,\mathbf{k}.$$

$$r'(t) = 2t\,i + 2j + k$$

$$\|r'(t)\| = \sqrt{4t^2 + 5}$$

$$T(t) = \frac{1}{4t^2+5}\left(2t\sqrt{4t^2+5}\,i + 2\sqrt{4t^2+5}\,j + \sqrt{4t^2+5}\,k\right)$$