Integrals of Vector-Valued Functions

limit, derivative, integral

componentwise

Definition

Let f, g, and h be integrable real-valued functions over the closed interval $[a, b]$.

1. The **indefinite integral of a vector-valued function** $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$ is

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j}. \tag{3.7}$$

The **definite integral of a vector-valued function** is

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j}]dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j}. \tag{3.8}$$

2. The indefinite integral of a vector-valued function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is

$$\int [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]dt = \left[\int f(t)dt\right]\mathbf{i} + \left[\int g(t)dt\right]\mathbf{j} + \left[\int h(t)dt\right]\mathbf{k}. \tag{3.9}$$

The definite integral of the vector-valued function is

$$\int_a^b [f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}]dt = \left[\int_a^b f(t)dt\right]\mathbf{i} + \left[\int_a^b g(t)dt\right]\mathbf{j} + \left[\int_a^b h(t)dt\right]\mathbf{k}. \tag{3.10}$$

3.8 Calculate the following integral:

$$\int_1^3 \left[(2t+4)\mathbf{i} + \left(3t^2 - 4t\right)\mathbf{j}\right]dt.$$

$$= \int_1^3 (2t+4)\,dt \; i + \int_1^3 (3t^2 - 4t)\,dt \; j$$

$$= \left(t^2 + 4t\right)\Big|_1^3 \, i + \left(t^3 - 2t^2\right)\Big|_1^3 \, j = (21-5)\,i + (9-(-1))\,j$$

$$= 16i + 10j$$

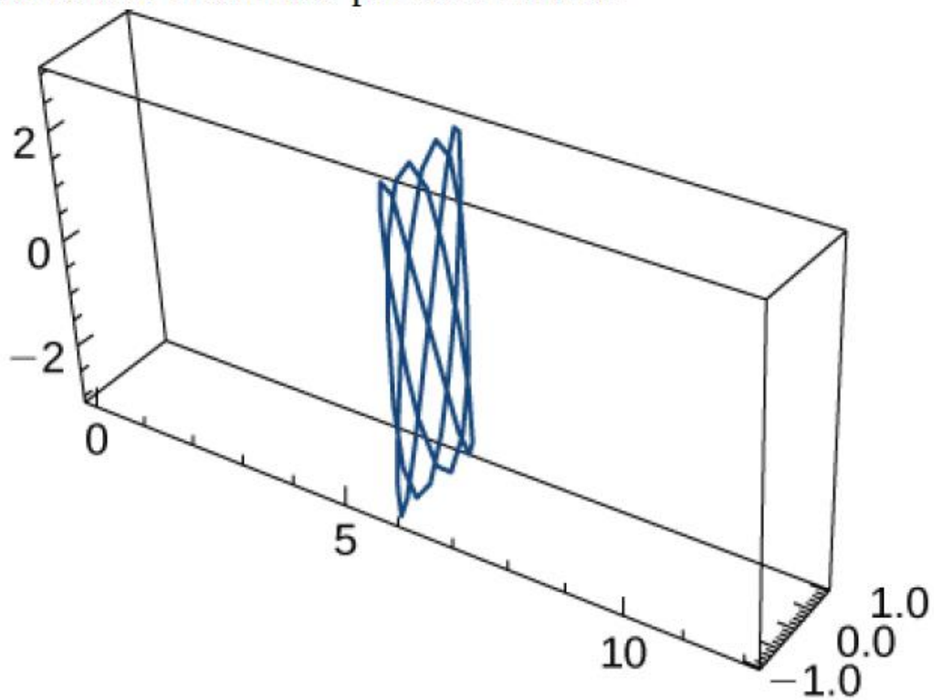
$$= \langle 16, 10 \rangle$$

$$\int (i + tj + e^t k)\,dt = (t + c_1)\,i + \left(\frac{t^2}{2} + c_2\right) j + (e^t + c_3)\,k$$

$$= \left\langle t, \frac{t^2}{2}, e^t \right\rangle + \langle c_1, c_2, c_3 \rangle$$

Find the unit tangent vector for the following parameterized curves.

56. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sin t\,\mathbf{k}, \quad 0 \le t < 2\pi.$ Two views of this curve are presented here:



$$r'(t) = -\sin t\,i + \cos t\,j + \cos t\,k$$

$$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \cos^2 t}$$

$$T(t) = \frac{1}{\sqrt{1+\cos^2 t}}\left(-\sin t\,i + \cos t\,j + \cos t\,k\right)$$

63. Find $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ for $\mathbf{r}(t) = -3t^5\,\mathbf{i} + 5t\,\mathbf{j} + 2t^2\,\mathbf{k}$.

$$r'(t) = \langle -15t^4, 5, 4t \rangle$$

$$r''(t) = \langle -60t^3, 0, 4 \rangle$$

$$r'(t) \cdot r''(t) = 900t^7 + \cancel{5 \cdot 0} + 16t$$ scalar function.

70. Find the antiderivative of $\mathbf{r}'(t) = \cos(2t)\,\mathbf{i} - 2\sin t\,\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$ that satisfies the initial condition $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

$r(t) = ?$

$$r(t) = \int r'(t)\,dt = \left(\frac{\sin 2t}{2} + c_1\right)i + \left(2\cos t + c_2\right)j + \left(\arctan t + c_3\right)k$$

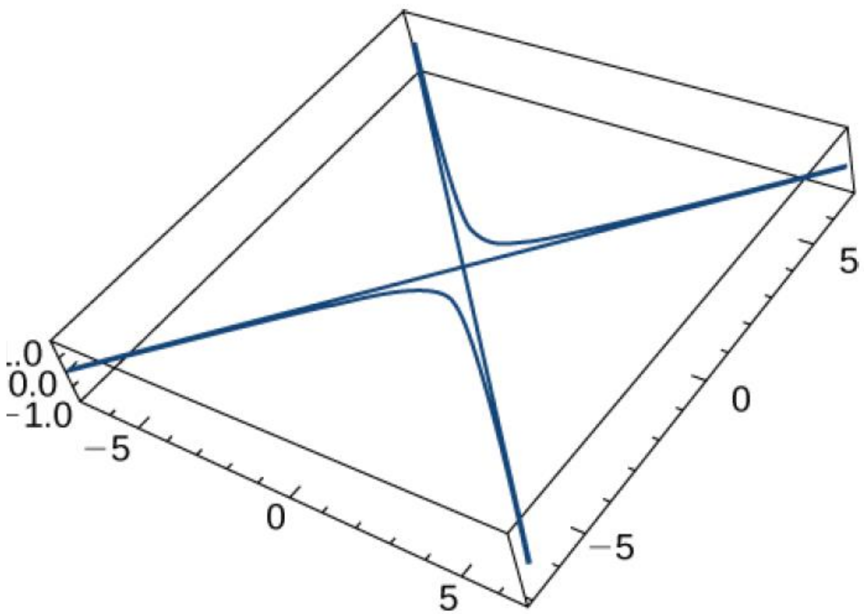
$$r(0) = 3i - 2j + k = c_1 i + (2 + c_2)j + c_3 k$$

$c_1 = 3$ $\quad -2 = 2 + c_2$ $\quad c_3 = 1$

$c_2 = -4$

$$r(t) = \left\langle \frac{\sin 2t}{2} + 3,\ 2\cos t - 4,\ \arctan t + 1 \right\rangle$$

Given the vector-valued function $\mathbf{r}(t) = \langle \tan t, \sec t, 0 \rangle$ (graph is shown here), find the following:



90. Velocity $r'(t) = \langle \sec^2 t, \sec t \tan t, 0 \rangle$

91. Speed $\|r'(t)\| = \sqrt{\sec^2 t(\sec^2 t + \tan^2 t)}$

92. Acceleration

$a(t) = v'(t) = r''(t)$

$= \langle 2\sec^2 t \tan t, \sec t \tan^2 t + \sec^3 t, 0 \rangle$

$\sec^2 t = 1 + \tan^2 t$

$y^2 = 1 + x^2$

$y^2 - x^2 = 1$ hyperbola

Evaluate the following integrals:

100. $\int \left(e^t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{1}{2t-1}\mathbf{k} \right) dt$ $= \left\langle e^t + c_1, -\cos t + c_2, \frac{\ln|2t-1|}{2} + c_3 \right\rangle$

101. $\int_0^1 \mathbf{r}(t)dt,$ where $\mathbf{r}(t) = \langle \sqrt[3]{t}, \frac{1}{t+1}, e^{-t} \rangle$

$$= \left\langle \frac{t^{4/3}}{4/3}\Big|_0^1, \ln|t+1|\Big|_0^1, -e^{-t}\Big|_0^1 \right\rangle$$

$$= \left\langle \frac{3}{4}, \ln 2, -\frac{1}{e} + 1 \right\rangle$$

Theorem 3.4: Arc-Length Formulas

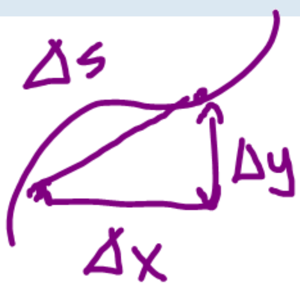
i. *Plane curve*: Given a smooth curve C defined by the function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$r'(t) = f'(t)i + g'(t)j$

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\,dt = \int_a^b \|\,\mathbf{r}'(t)\,\|\;dt. \tag{3.11}$$

ii. *Space curve*: Given a smooth curve C defined by the function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where t lies within the interval $[a, b]$, the arc length of C over the interval is

$$s = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt = \int_a^b \|\,\mathbf{r}'(t)\,\|\;dt. \tag{3.12}$$



$$S = \int_a^b ds = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$\hookrightarrow \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \cdot dt$$

3.9 Calculate the arc length of the parameterized curve

$$\mathbf{r}(t) = \langle\, 2t^2+1,\ 2t^2-1,\ t^3 \,\rangle\ ,\ 0 \le t \le 3.$$

$$r'(t) = \langle 4t,\ 4t,\ 3t^2 \rangle$$

$$S = \int_0^3 \|r'(t)\|\,dt = \int_0^3 \sqrt{32t^2+9t^4}\,dt = \int_0^3 t\sqrt{32+9t^2}\,dt$$

$u = 32+9t^2$

$du = 18t\,dt$

$t=3,\ u=32+81$

$$\int_{32}^{113} u^{1/2}\,\frac{du}{18} = \frac{1}{18}\,\frac{u^{3/2}}{3/2}\Big|_{32}^{113} = \frac{1}{27}\left(113^{3/2} - 2^{15/2}\right)$$

$$= \frac{1}{27}\left(113\sqrt{113} - 128\sqrt{2}\right)$$

Theorem 3.5: Arc-Length Function

Let $\mathbf{r}(t)$ describe a smooth curve for $t \geq a$. Then the arc-length function is given by

$$s(t) = \int_a^t \| \mathbf{r}'(u) \| \, du. \tag{3.14}$$

Furthermore, $\frac{ds}{dt} = \| \mathbf{r}'(t) \| > 0$. If $\| \mathbf{r}'(t) \| = 1$ for all $t \geq a$, then the parameter t represents the arc length from the starting point at $t = a$.

arc-length parametrization

3.10 Find the arc-length function for the helix

$$\mathbf{r}(t) = \langle 3\cos t,\ 3\sin t,\ 4t \rangle,\ t \geq 0.$$

Then, use the relationship between the arc length and the parameter t to find an arc-length parameterization of $\mathbf{r}(t)$.

$r'(u) = \langle -3\sin u, 3\cos u, 4 \rangle$

$\|r'(u)\| = \sqrt{9(\sin^2 u + \cos^2 u) + 4^2}$

$\|r'(u)\| = \sqrt{25} = 5$

$s(t) = \int_0^t 5\,du = 5u\Big|_0^t = 5t$

$s = 5t$ $\quad t = \frac{s}{5}$

arc-length parametrization

$r(s) = \langle 3\cos\frac{s}{5}, 3\sin\frac{s}{5}, \frac{4s}{5} \rangle$

$\|r'(s)\| = 1$ $\quad r'(s) = \langle -\frac{3}{5}\sin\frac{s}{5}, \frac{3}{5}\cos\frac{s}{5}, \frac{4}{5} \rangle$

Definition

Let C be a smooth curve in the plane or in space given by $\mathbf{r}(s)$, where s is the arc-length parameter. The **curvature** κ at s is

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \| \mathbf{T}'(s) \| .$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\| \mathbf{r}'(t) \|}$$

unit tangent vector

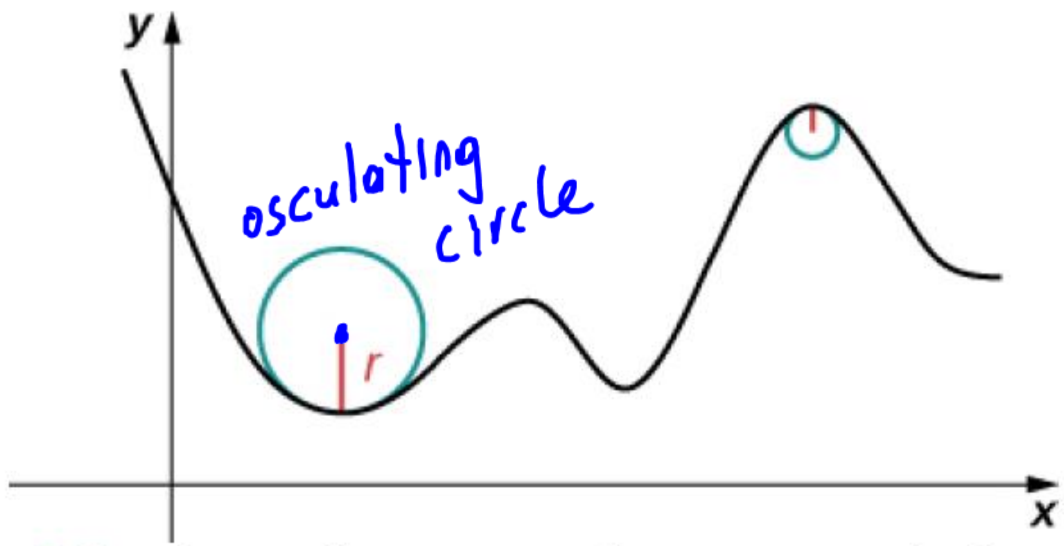


Figure 3.6 The graph represents the curvature of a function $y = f(x)$. The sharper the turn in the graph, the greater the curvature, and the smaller the radius of the inscribed circle.

Theorem 3.6: Alternative Formulas for Curvature

If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature κ of C at t is given by

$$\kappa = \frac{\| \mathbf{T}'(t) \|}{\| \mathbf{r}'(t) \|}. \tag{3.15}$$

If C is a three-dimensional curve, then the curvature can be given by the formula

$$\kappa = \frac{\| \mathbf{r}'(t) \times \mathbf{r}''(t) \|}{\| \mathbf{r}'(t) \|^3}. \tag{3.16}$$

If C is the graph of a function $y = f(x)$ and both y' and y'' exist, then the curvature κ at point (x, y) is given by

$$\kappa = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}. \tag{3.17}$$

3.11 Find the curvature of the curve defined by the function

$$y = 3x^2 - 2x + 4$$

at the point $x = 2$.

$$y' = 6x - 2$$

$$y'' = 6$$

$$K = \frac{|y''|}{(1+(y')^2)^{3/2}}$$

at $x = 2$

$$K = \frac{6}{(1+(6x-2)^2)^{3/2}}$$

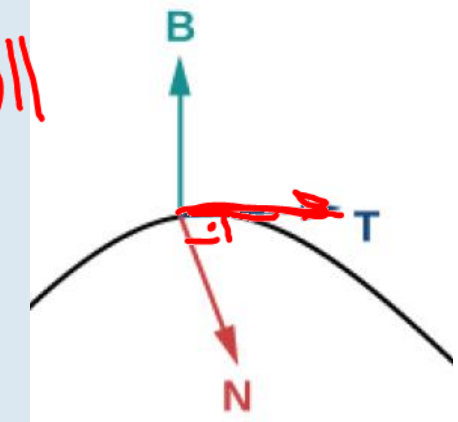
$$K = \frac{6}{(1+10^2)^{3/2}} = \frac{6}{101\sqrt{101}}$$

Definition

Let C be a three-dimensional **smooth** curve represented by $\mathbf{r}$ over an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the principal unit normal vector at t is defined to be

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\| \mathbf{T}'(t) \|}.$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$



The binormal vector at t is defined as

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t),$$

where $\mathbf{T}(t)$ is the unit tangent vector.

3.12 Find the unit normal vector for the vector-valued function $\mathbf{r}(t) = \left(t^2 - 3t\right)\mathbf{i} + (4t+1)\,\mathbf{j}$ and evaluate it at $t = 2$.

$$T(t) = \frac{\langle 2t-3,\ 4\rangle}{\sqrt{(2t-3)^2+4^2}} = \left\langle \frac{2t-3}{\sqrt{(2t-3)^2+16}},\ \frac{4}{\sqrt{(2t-3)^2+16}} \right\rangle$$

$$T'(t) = \left\langle \frac{2\sqrt{(2t-3)^2+16} + (2t-3)\cdot\frac{\cancel{2}(2t-3)}{\cancel{2}\sqrt{(2t-3)^2+16}}}{(2t-3)^2+16},\ 4\,\frac{2(2t-3)}{2\sqrt{(2t-3)^2+16}} \right\rangle$$

$$\frac{2\left[(2t-3)^2+16\right]+(2t-3)^2}{\left((2t-3)^2+16\right)^{3/2}} \ldots\ldots$$

3.13 Find the equation of the osculating circle of the curve defined by the vector-valued function $y = 2x^2 - 4x + 5$ at $x = 1$.

K at $x=1$ is 4

$$K = \frac{|y''|}{(1+(y')^2)^{3/2}} = \frac{4}{(1+(4x-4)^2)^{3/2}}$$

$K = 4$

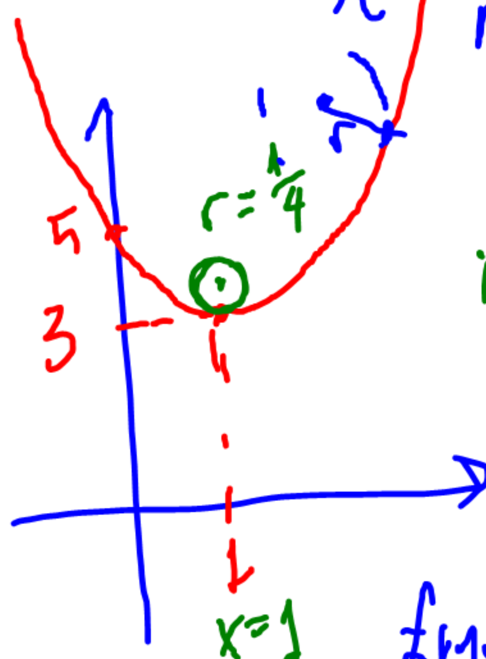
$K = \frac{1}{r}$ $\quad Kr = 1 \quad r = \frac{1}{K} = \frac{1}{4}$

$y' = 4x - 4$

$y'' = 4$

$$-\frac{b}{2a} = -\frac{-4}{2 \cdot 2} = 1$$

$f(1) = 2 - 4 + 5 = 3$



center of the circle is $\left(1, 3 + \frac{1}{4}\right)$

$$(x-1)^2 + \left(y - \frac{13}{4}\right)^2 = \frac{1}{16}$$

finding the center is more challenging for other points