5.6 | Calculating Centers of Mass and Moments of Inertia

Learning Objectives

5.6.1 Use double integrals to locate the center of mass of a two-dimensional object.
5.6.2 Use double integrals to find the moment of inertia of a two-dimensional object.
5.6.3 Use triple integrals to locate the center of mass of a three-dimensional object.

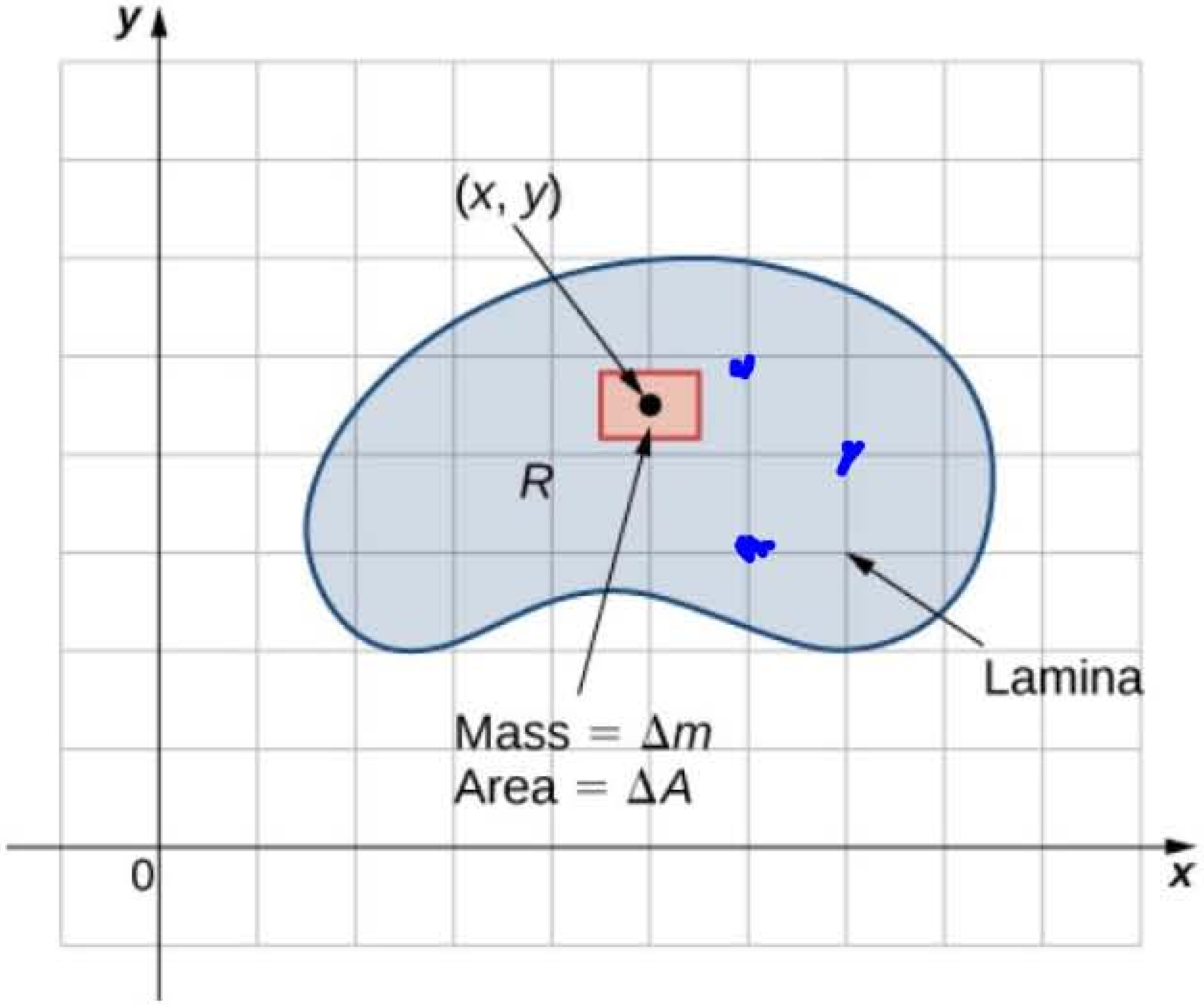


Figure 5.65 The density of a lamina at a point is the limit of its mass per area in a small rectangle about the point as the area goes to zero.

Hence, the mass of the lamina is $M = \iint_R \underbrace{\rho(x, y)}_{\text{density}} dA.$

mass = density × volume (area in 2D)

$\Delta m = \rho(x,y)\,\Delta A$

$dm = \rho(x,y)\,dA$

$M = \iint_R dm = \iint_R \rho(x,y)\,dA$

Slide 1

5.33 Consider the same region R as in the previous example, and use the density function $\rho(x, y) = \sqrt{xy}$. Find the total mass.

Type I

$0 \leq x \leq 3$

$0 \leq y \leq 3-x$

$$\iint_R \rho(x,y)\,dA$$

$$\int_0^3 \int_0^{3-x} \sqrt{xy}\,dy\,dx$$

$$\int_0^3 \sqrt{x}\, \frac{y^{3/2}}{3/2} \Big|_0^{3-x} dx = \int_0^3 \sqrt{x}\,(3-x)^{3/2}\, \frac{2}{3}\, dx$$

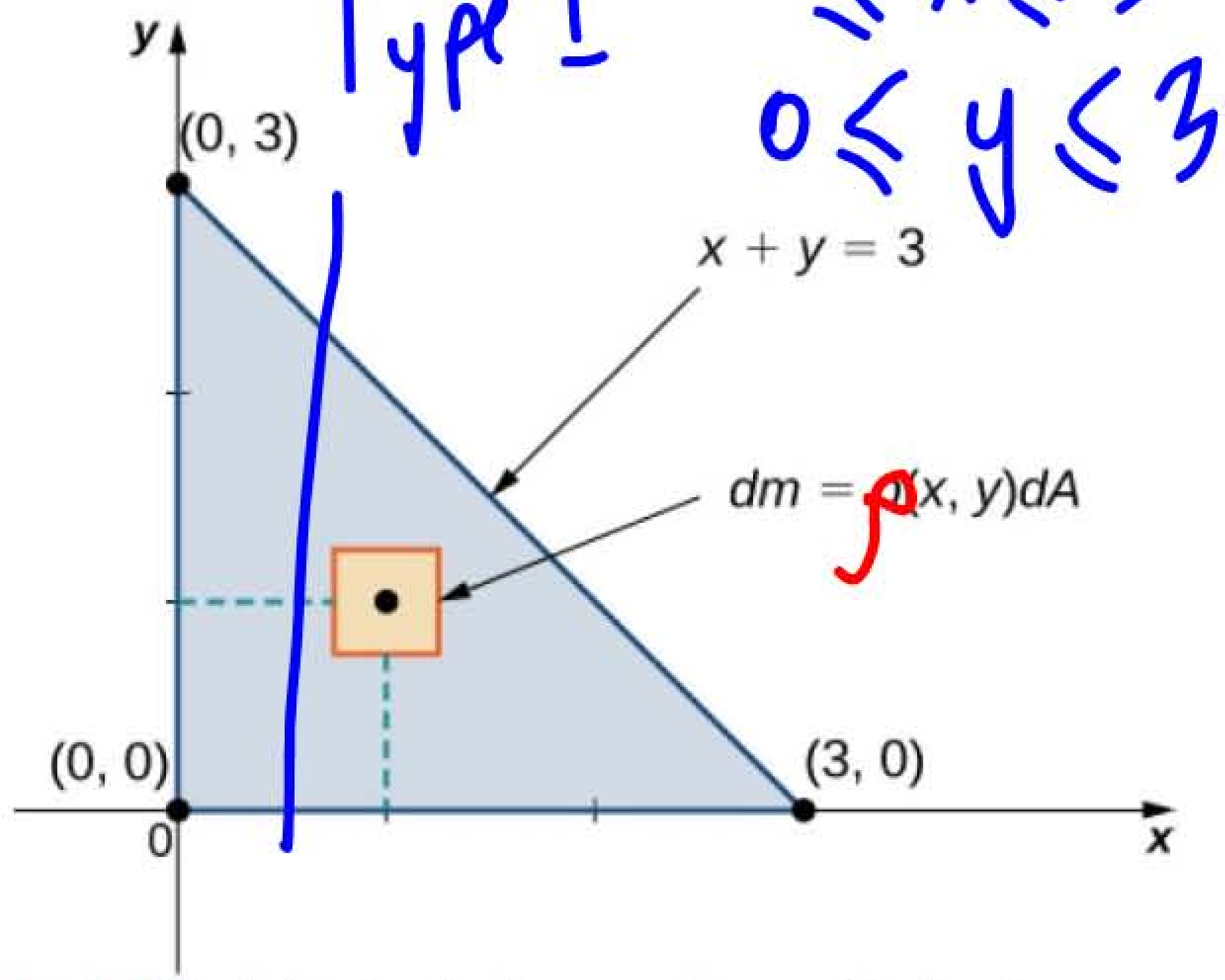


Figure 5.67 A lamina in the xy-plane with density $\rho(x, y) = \sqrt{xy}$.

$$M = \text{force} \times \text{distance}$$

$$\Delta M = dm \times y = y\rho(x, y)\,dA$$

The moment M_x about the x-axis for R is

$$\boxed{M_x = \iint_R y\rho(x, y)\,dA.}$$

Similarly, the moment M_y about the y-axis for R is

$$\boxed{M_y = \iint_R x\rho(x, y)\,dA.}$$

the center of mass

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\rho(x, y)\,dA}{\iint_R \rho(x, y)\,dA} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{\iint_R y\rho(x, y)\,dA}{\iint_R \rho(x, y)\,dA}.$$

5.34 Consider the same lamina R as above, and use the density function ~~$\rho(x, y) = \sqrt{xy}$~~. Find the moments M_x and M_y.

$$\rho(x,y) = xy$$

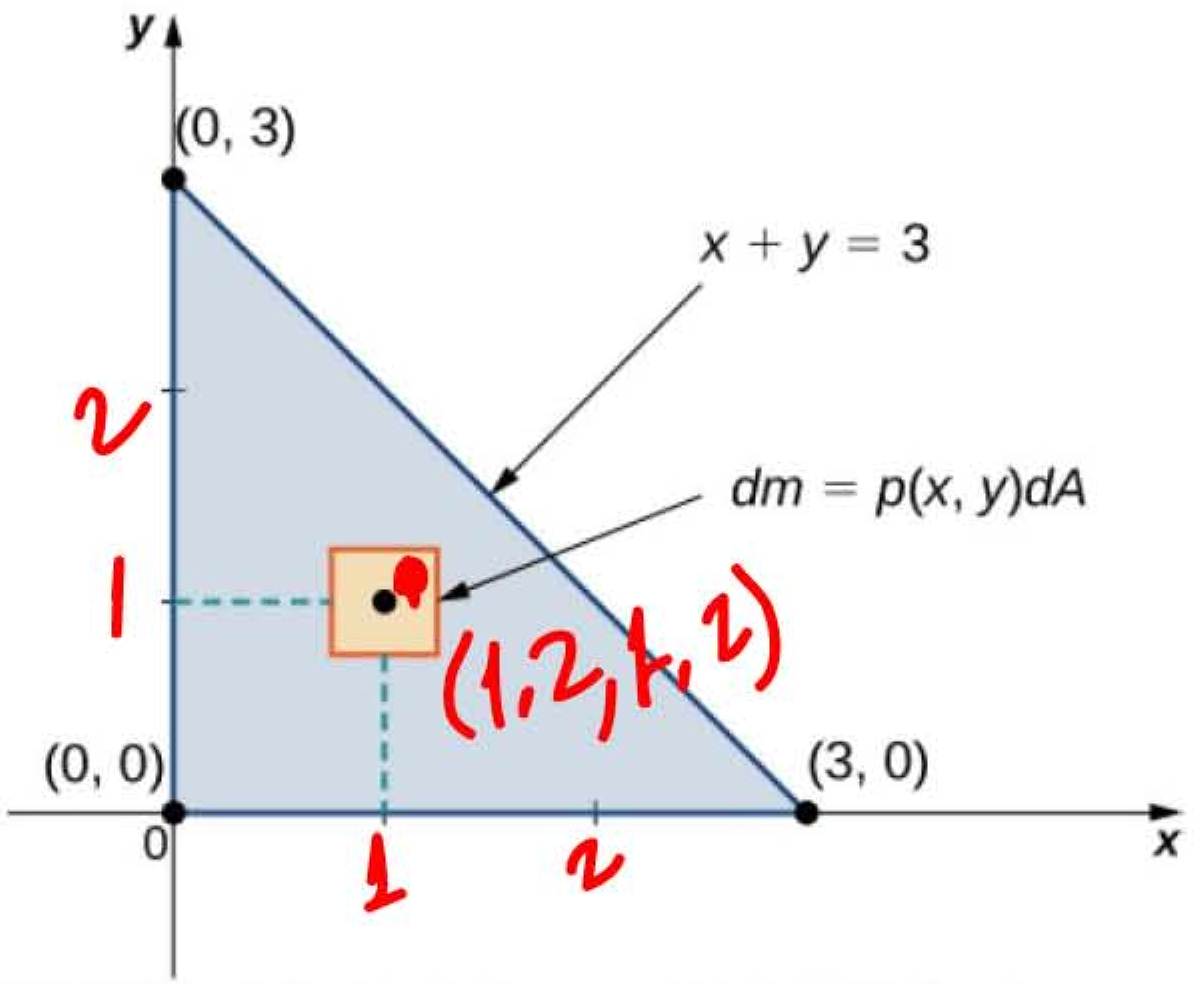


Figure 5.67 A lamina in the xy-plane with density

$$M_x = \int_0^3 \int_0^{3-x} y \cdot xy \, dy\, dx$$

$$= \int_0^3 x \left. \frac{y^3}{3} \right|_0^{3-x} dx = \int_0^3 \frac{x(3-x)^3}{3} dx$$

$u = 3 - x$

$du = -dx$

$x = 3 - u$

$$\int_3^0 (3-u)\frac{u^3}{3}(-du) = \int_3^0 \left(-u^3 + \frac{u^4}{3}\right) du = \left.\left(-\frac{u^4}{4} + \frac{u^5}{15}\right)\right|_3^0$$

$$= 0 - \left(-\frac{81}{4} + \frac{81}{5}\right) = \frac{81}{20}$$

$$M_y = \int_0^3 \int_0^{3-x} x \cdot xy \, dy \, dx = \int_0^3 x^2 \frac{y^2}{2} \Big|_0^{3-x} dx$$

$$= \int_0^3 x^2 \frac{(3-x)^2}{2} dx = \int_0^3 \frac{x^2(9-6x+x^2)}{2} dx = \frac{1}{2} \int_0^3 \left(x^4 - 6x^3 + 9x^2\right) dx$$

$$= \frac{1}{2}\left(\frac{x^5}{5} - \frac{3x^4}{2} + 3x^3\right)\Big|_0^3 = \frac{1}{2}\left(\underset{(2)}{\frac{243}{5}} - \underset{(5)}{\frac{243}{2}} + 81\right) = \frac{1}{2}\left(\frac{10 \times 81 + 2 \times 243 - 5 \times 243}{10}\right)$$

$$= \frac{1}{2} \frac{(10+6-15) \times 81}{10} = \underline{\frac{81}{20}}$$

5.35 Again use the same region R as above and the density function $\rho(x, y) = \cancel{\sqrt{xy}}$ xy. Find the center of mass.

$$M = \iint_R \rho(x,y)\,dA = \int_0^3 \int_0^{3-x} xy\,dy\,dx = \int_0^3 x \cdot \frac{y^2}{2}\Big|_0^{3-x} dx$$

$$= \int_0^3 \frac{x(3-x)^2}{2} dx = \int_3^0 \frac{(3-u)u^2}{2}(-du) = \int_0^3 \left(\frac{3u^2}{2} - \frac{u^3}{2}\right) du$$

$u = 3-x$
$x = 3-u$
$du = -dx$

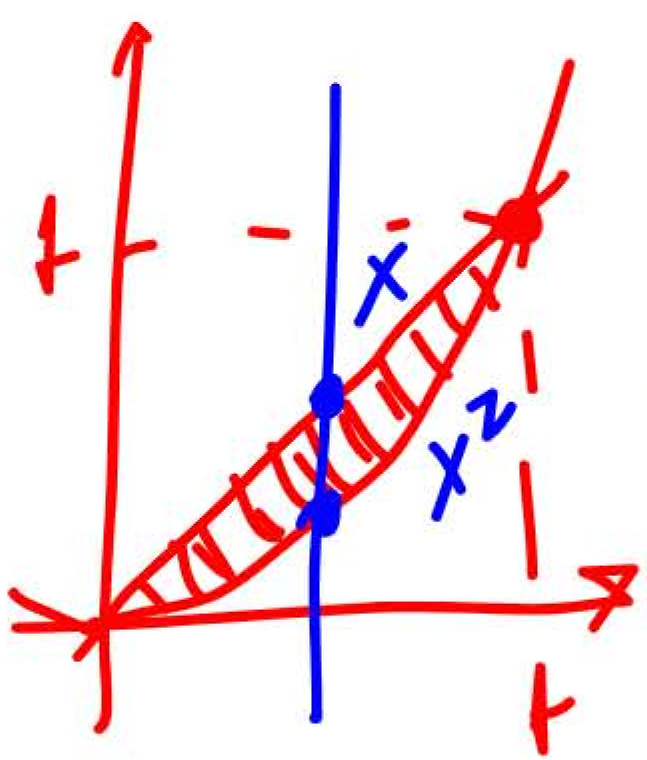
$$= \left(\frac{u^3}{2} - \frac{u^4}{8}\right)\Big|_0^3 = \frac{27}{2}_{(4)} - \frac{81}{8} = \frac{27}{8}.$$ mass

$$\bar{x} = \frac{M_y}{M} = \frac{81/20}{27/8} = \frac{24}{20} = \frac{6}{5}$$

$$\bar{y} = \frac{6}{5}$$

Center of mass $(1.2, 1.2)$

5.36 Calculate the mass, moments, and the center of mass of the region between the curves $y = x$ and $y = x^2$ with the density function $\rho(x, y) = x$ in the interval $0 \le x \le 1$.



$0 \le x \le 1$

$x^2 \le y \le x$

$$m = \iint_R \rho(x,y)\, dA = \int_0^1 \int_{x^2}^{x} x \, dy\, dx = \int_0^1 xy \Big|_{x^2}^{x} dx$$

mass

$$= \int_0^1 (x^2 - x^3)\, dx = \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} = m$$

$$M_x = \int_0^1 \int_{x^2}^{x} y x \, dy\, dx = \int_0^1 \frac{y^2}{2} x \Big|_{x^2}^{x} dx = \int_0^1 \left(\frac{x^3}{2} - \frac{x^5}{2}\right) dx = \left(\frac{x^4}{8} - \frac{x^6}{12}\right)\Big|_0^1$$

$$= \frac{1}{8} - \frac{1}{12} = \frac{12-8}{96} = \frac{4}{96}$$

$$M_x = \frac{1}{24}$$

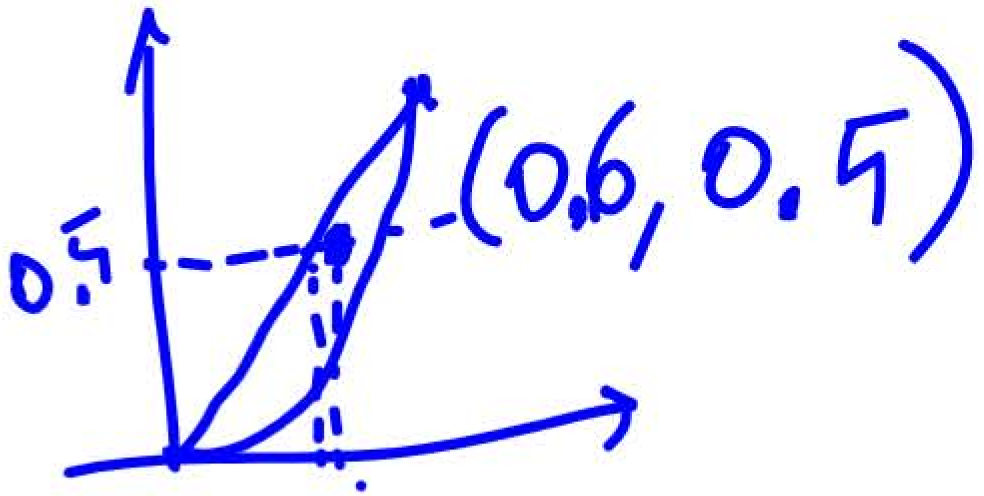
$$M_y = \int_0^1 \int_{x^2}^{x} x \cdot x \, dy\, dx = \int_0^1 x^2 y \Big|_{x^2}^{x} dx = \int_0^1 (x^3 - x^4)\, dx$$

$$= \left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

center of mass

$$\bar{x} = \frac{M_y}{m} = \frac{1/20}{1/12} = \frac{12}{20} = \frac{3}{5}$$

$$\bar{y} = \frac{M_x}{m} = \frac{1/24}{1/12} = \frac{1}{2} = 0.5$$



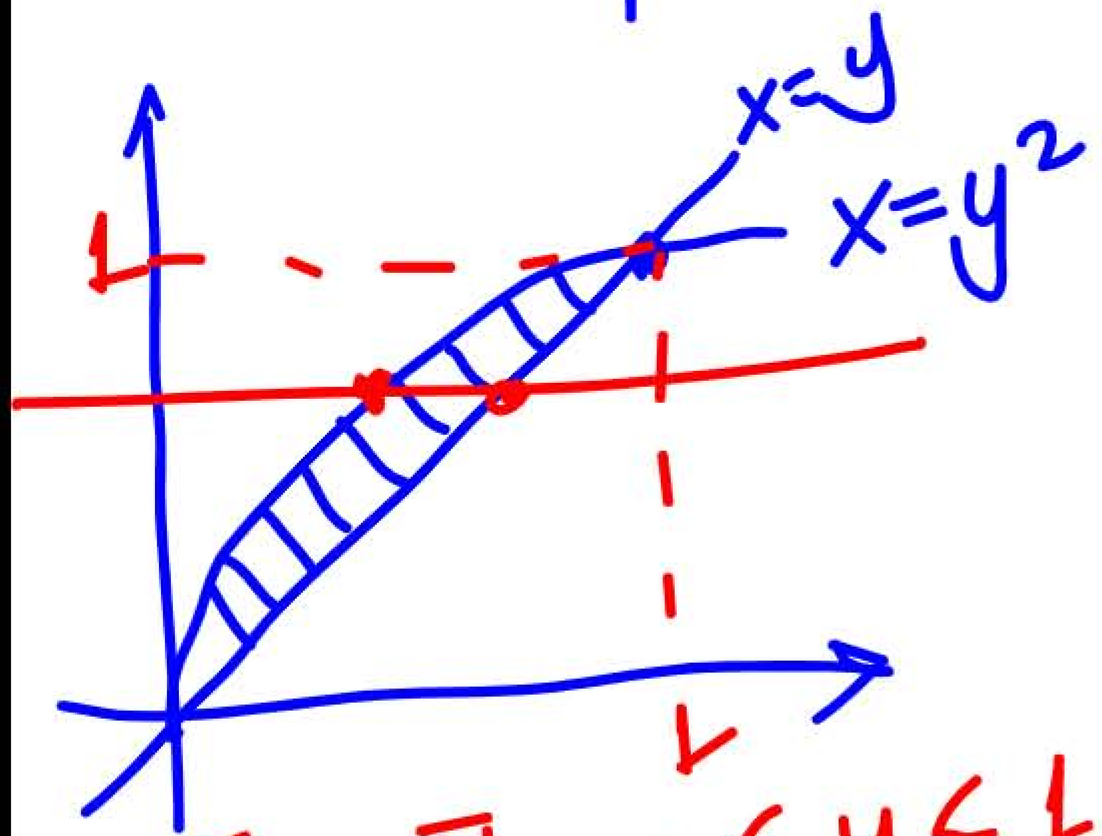
5.37 Calculate the centroid of the region between the curves $y = x$ and $y = \sqrt{x}$ with uniform density in the interval $0 \le x \le 1$.

density is constant – ρ

$$\bar{X} = \frac{M_y}{m} = \frac{\iint_R x \cdot \rho \, dA}{\iint_R \rho \, dA} = \frac{\cancel{\rho}\iint_R x \, dA}{\cancel{\rho}\iint_R dA}$$

$$m = \int_0^1 \int_{y^2}^{y} dx\,dy = \int_0^1 (y - y^2)\,dy$$

$$= \left(\frac{y^2}{2} - \frac{y^3}{3}\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}\rho$$



Type II $\quad 0 \le y \le 1$

$y^2 \le x \le y$

$$M_y = \int_0^1 \int_{y^2}^{y} x \, dx\,dy = \int_0^1 \frac{x^2}{2}\Big|_{y^2}^{y} dy = \int_0^1 \left(\frac{y^2}{2} - \frac{y^4}{2}\right) dy$$

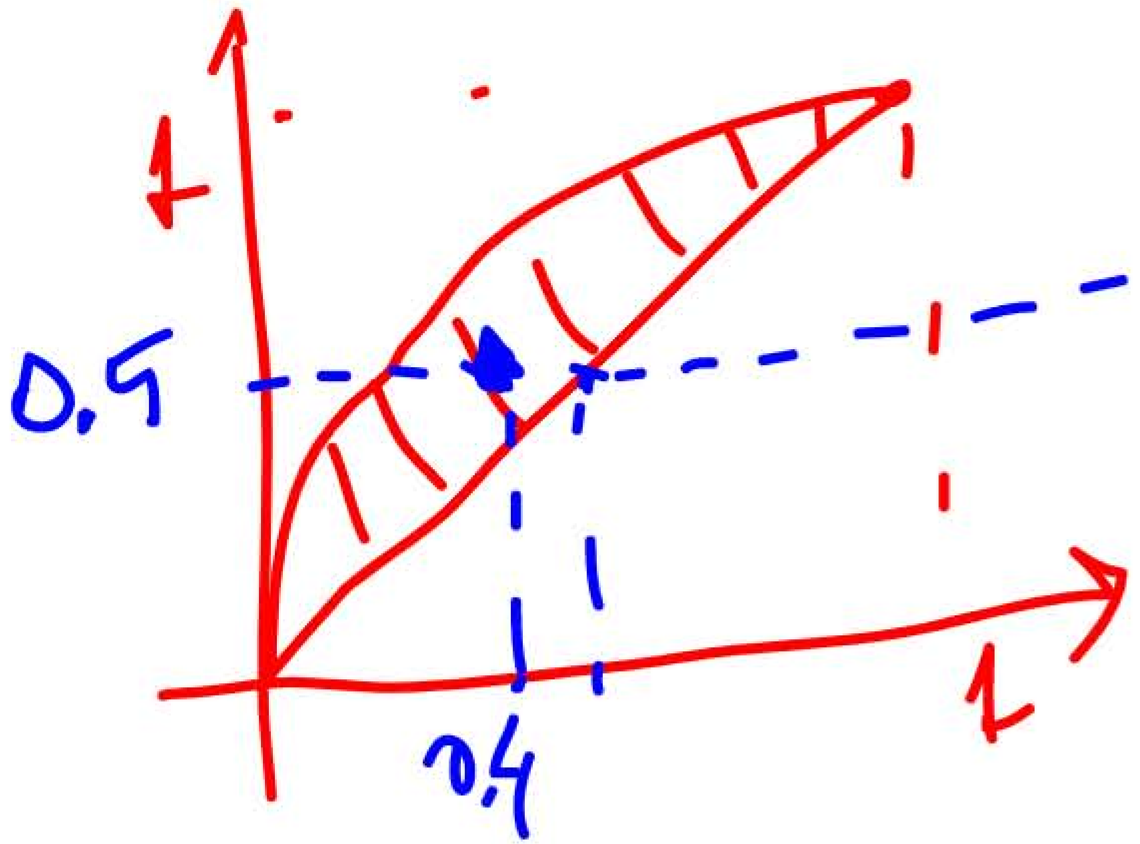
$$= \left(\frac{y^3}{6} - \frac{y^5}{10}\right)\Big|_0^1 = \frac{1}{6} - \frac{1}{10} = \frac{4}{60} = \frac{1}{15}\rho$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{1}{15}\rho}{\frac{1}{6}\rho} = \frac{6}{15} = \frac{2}{5} = 0.4$$

$$M_x = \int_0^1 \int_{y^2}^{y} y \, dx\, dy = \int_0^1 yx \Big|_{y^2}^{y} dy = \int_0^1 (y^2 - y^3)\, dy = \left(\frac{y^3}{3} - \frac{y^4}{4}\right)\Big|_0^1 = \frac{1}{12}\rho$$

centeroid is (0.4, 0.5)

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{1}{12}\rho}{\frac{1}{6}\rho} = \frac{1}{2} = 0.5$$



The moment of inertia I_x about the x-axis for the region R is the limit of the sum of moments of inertia of the regions R_{ij} about the x-axis. Hence

$M_x = \iint_R y\rho(x,y)\,dA$

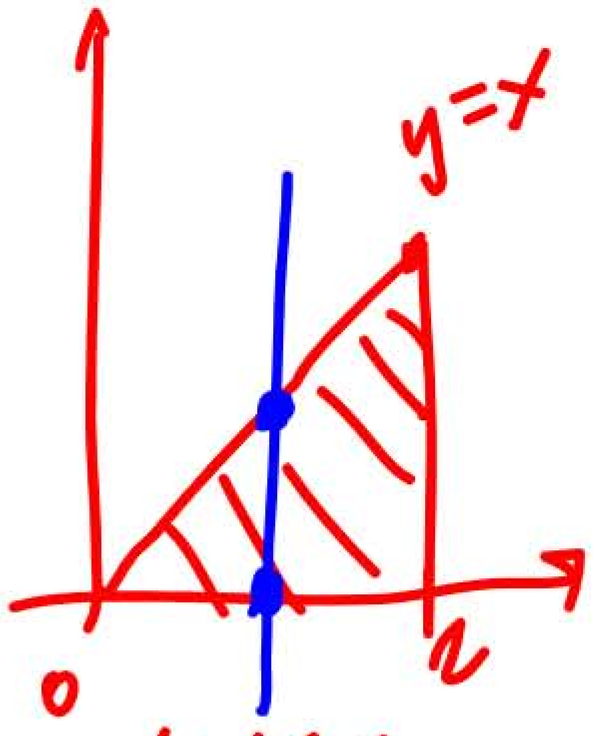
$$I_x = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^*\right)^2 m_{ij} = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(y_{ij}^*\right)^2 \rho\left(x_{ij}^*,\, y_{ij}^*\right)\Delta A = \iint_R y^2 \rho(x,\, y)dA.$$

Similarly, the moment of inertia I_y about the y-axis for R is the limit of the sum of moments of inertia of the regions R_{ij} about the y-axis. Hence

$$I_y = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(x_{ij}^*\right)^2 m_{ij} = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \left(x_{ij}^*\right)^2 \rho\left(x_{ij}^*,\, y_{ij}^*\right)\Delta A = \iint_R x^2 \rho(x,\, y)dA. = I_y$$

Use the triangular region R with vertices $(0, 0)$, $(2, 2)$, and $(2, 0)$

5.38 Again use the same region R as above and the density function $\rho(x, y) = \sqrt{xy}$. Find the moments of inertia.



$0 \le x \le 2$

$0 \le y \le x$

$$I_x = \int_0^2\int_0^x y^2 \cdot \sqrt{xy}\, dy\, dx = \int_0^2\int_0^x \sqrt{x}\, y^{5/2}\, dy\, dx$$

$$= \int_0^2 \sqrt{x} \left.\frac{y^{7/2}}{7/2}\right|_0^x dx = \int_0^2 \frac{2}{7} x^{\frac{1}{2}+\frac{7}{2}}\, dx = \left.\frac{2x^5}{7 \cdot 5}\right|_0^2 = \frac{64}{35}$$

$$I_y = \int_0^2\int_0^x x^2 \sqrt{xy}\, dy\, dx = \int_0^2 x^{5/2} \left.\frac{y^{3/2}}{3/2}\right|_0^x dx = \int_0^2 \frac{2}{3} x^4\, dx$$

$$= \frac{2}{3} \left.\frac{x^5}{5}\right|_0^2 = \frac{64}{15} = I_y$$

Center of Mass and Moments of Inertia in Three Dimensions

Definition

If we have a solid object Q with a density function $\rho(x, y, z)$ at any point (x, y, z) in space, then its mass is

mass

$$\boxed{m = \iiint_Q \rho(x, y, z)\,dV.}$$

$dm = \rho\, dV$, $m = \iiint_Q dm$

Its moments about the xy-plane, the xz-plane, and the yz-plane are

$$M_{xy} = \iiint_Q z\rho(x, y, z)\,dV, \quad M_{xz} = \iiint_Q y\rho(x, y, z)\,dV,$$

$$M_{yz} = \iiint_Q x\rho(x, y, z)\,dV.$$

If the center of mass of the object is the point $(\bar{x}, \bar{y}, \bar{z})$, then

$$\boxed{\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}.}$$

moments of inertia

$$I_x = \iiint_Q (y^2 + z^2)\rho(x, y, z)\,dV,$$

$$I_y = \iiint_Q (x^2 + z^2)\rho(x, y, z)\,dV,$$

$$I_z = \iiint_Q (x^2 + y^2)\rho(x, y, z)\,dV.$$

5.40 Consider the same region Q (**Figure 5.70**), and use the density function $\rho(x, y, z) = xy^2 z$. Find the mass.

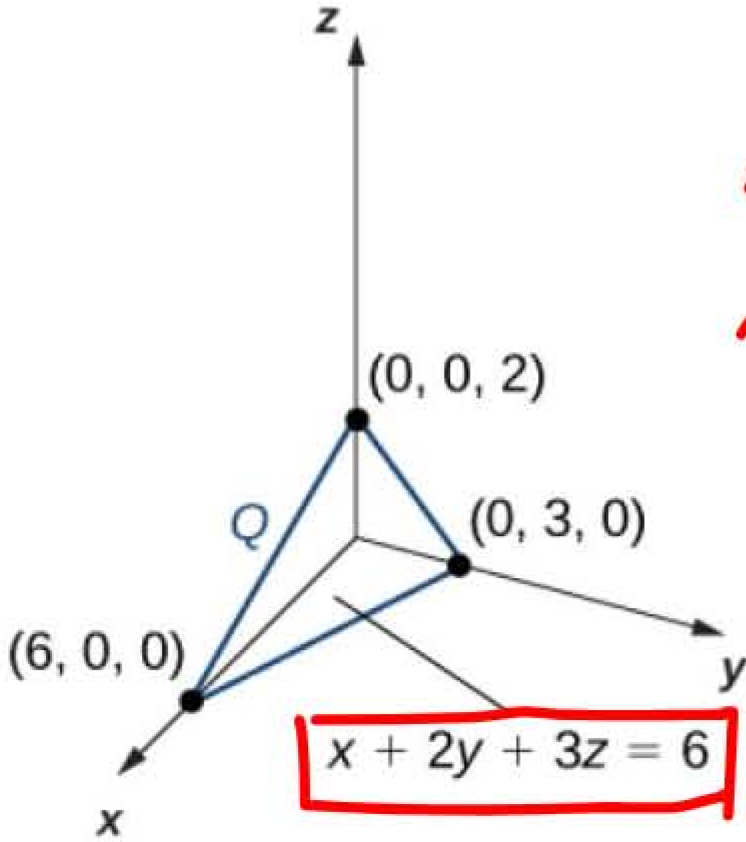


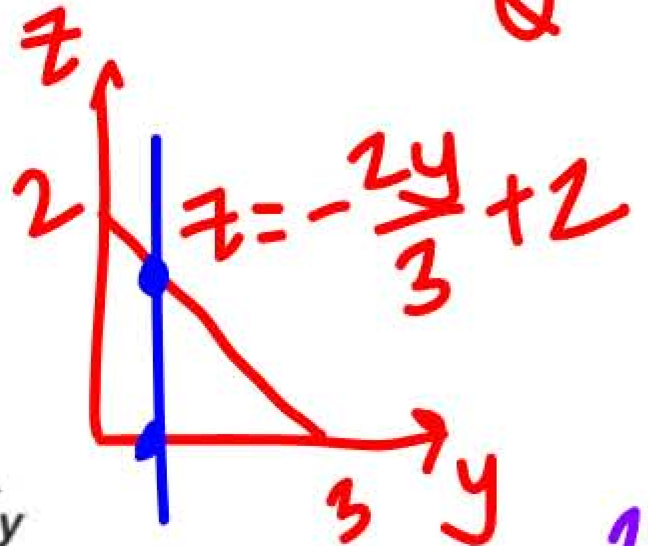
Figure 5.70 Finding the mass of a three-dimensional solid Q.

$x = 6 - 2y - 3z$

$0 \leq x \leq 6 - 2y - 3z$

$0 \leq z \leq -\frac{2y}{3} + 2$

$0 \leq y \leq 3$



$$m = \iiint_Q xy^2 z \, dV$$

$$\int_0^3 \int_0^{-\frac{2y}{3}+2} \int_0^{6-2y-3z} x y^2 z \, dx\, dz\, dy$$

$$\int_0^3 \int_0^{-\frac{2y}{3}+2} \frac{(6-2y-3z)^2}{2} y^2 z \, dz\, dy$$

$$\frac{(6-2y)^2 y^2 z}{2} - \frac{2(6-2y)3z \, y^2 z}{2} + \frac{(3z)^2 y^2 z}{2}$$