5.5 | Triple Integrals in Cylindrical and Spherical Coordinates

(r, θ, z)

(ρ, ϕ, θ)

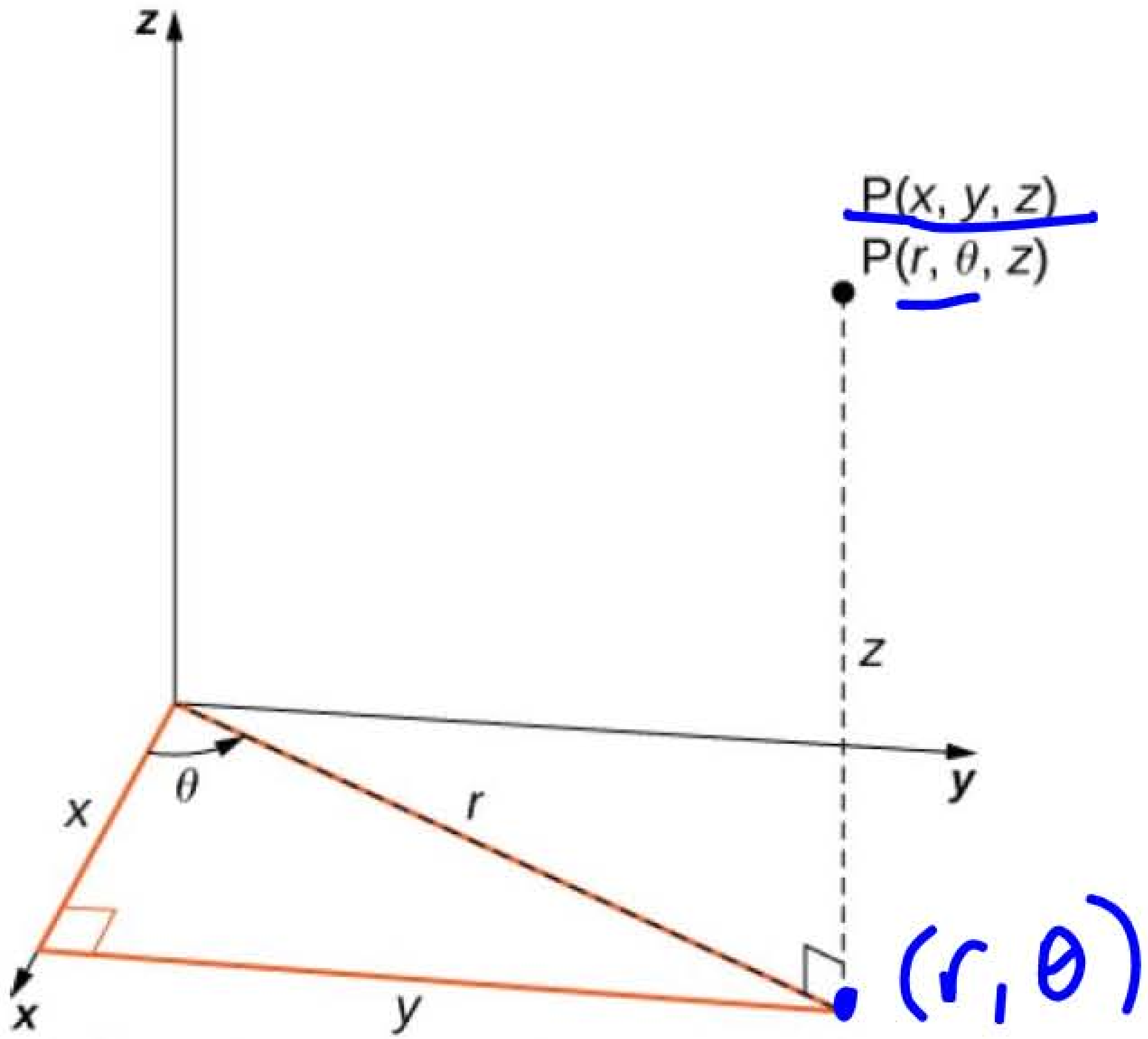

Figure 5.50 Cylindrical coordinates are similar to polar coordinates with a vertical z coordinate added.

$x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

$\tan\theta = \dfrac{y}{x}$

$dA = dx\,dy$

$= r\,dr\,d\theta$

$dV = dx\,dy\,dz$

$= r\,dr\,d\theta\,dz$

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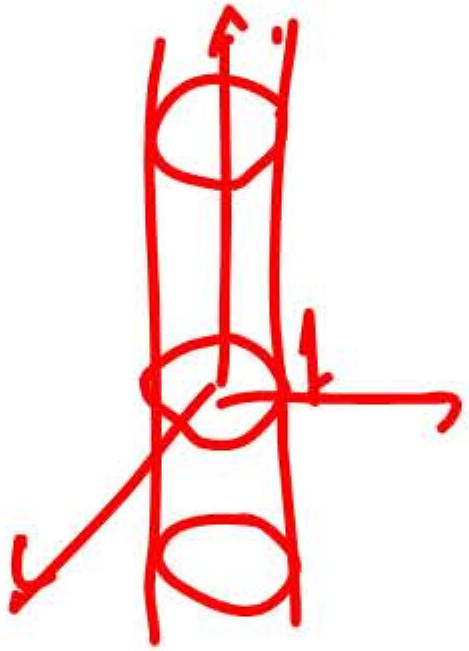
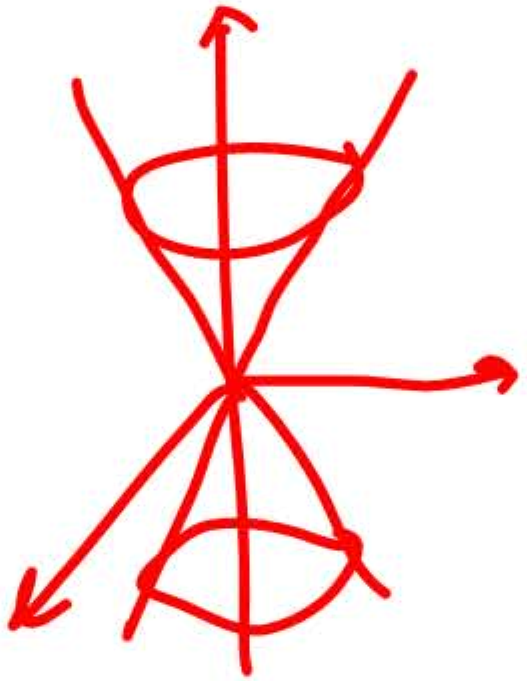
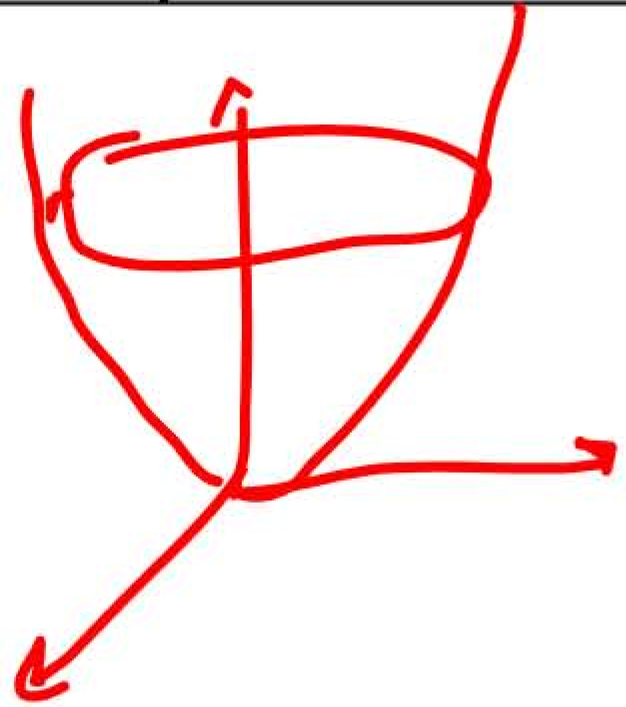
	Circular cylinder	Circular cone	Sphere	Paraboloid
Rectangular	$x^2 + y^2 = c^2$	$z^2 = c^2\left(x^2 + y^2\right)$	$x^2 + y^2 + z^2 = c^2$	$z = c\left(x^2 + y^2\right)$
Cylindrical	$r = c$	$z = cr$	$r^2 + z^2 = c^2$	$z = cr^2$

Table 5.1 Equations of Some Common Shapes







Theorem 5.12: Fubini's Theorem in Cylindrical Coordinates

Suppose that $g(x, y, z)$ is continuous on a rectangular box B, which when described in cylindrical coordinates looks like $B = \{(r, \theta, z) | a \leq r \leq b, \alpha \leq \theta \leq \beta, c \leq z \leq d\}$.

Then $g(x, y, z) = g(r\cos\theta, r\sin\theta, z) = f(r, \theta, z)$ and

$$\iiint\limits_B g(x, y, z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(r, \theta, z)r\,dr\,d\theta\,dz.$$

$$\int_0^4 \int_0^\pi \int_1^2 f(r,\theta,z)\,r\,dr\,d\theta\,dz$$

$0 \leq \theta \leq \pi$

$1 \leq r \leq 2$

5.27

Evaluate the triple integral $\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=1}\int_{z=0}^{z=4} rz\sin\theta r\,dz\,dr\,d\theta$

$$r^2\sin\theta\frac{z^2}{2}\Big|_0^4$$

$$\int_0^{\pi}\int_0^{1} 8r^2\sin\theta\,dr\,d\theta$$

$$\int_0^{\pi}\frac{8r^3}{3}\sin\theta\Big|_0^1 d\theta$$

$\cos\pi=-1$

$\cos 0=1$

$0\leqslant\theta\leqslant\pi$

$0\leqslant r\leqslant 1$

$0\leqslant z\leqslant 4$

$$=\int_0^{\pi}\frac{8}{3}\sin\theta\,d\theta=-\frac{8}{3}\cos\theta\Big|_0^{\pi}$$

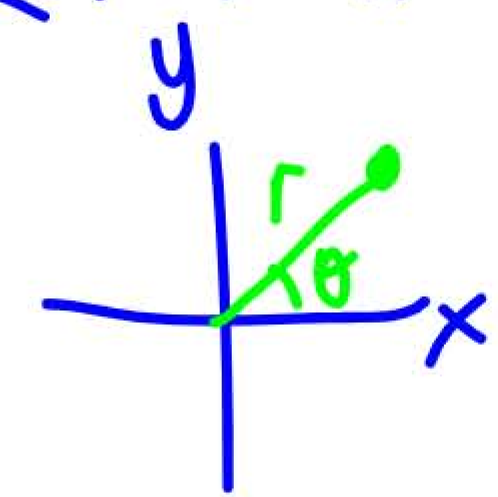
$$=-\frac{8}{3}(-1-1)=\frac{16}{3}$$

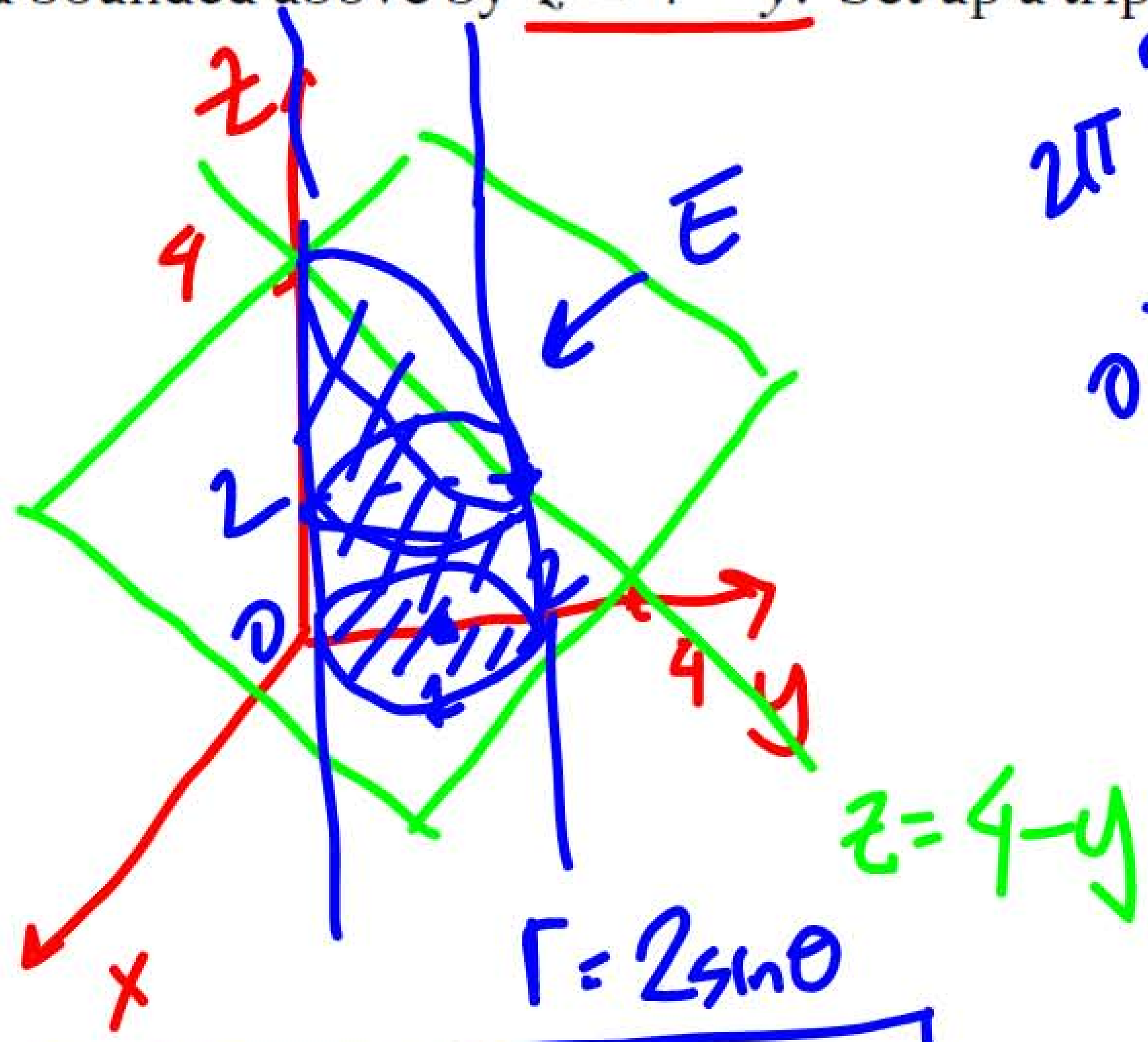
5.28 Consider the region E inside the right circular cylinder with equation $r = 2\sin\theta$, bounded below by the $r\theta$-plane and bounded above by $z = 4 - y$. Set up a triple integral with a function $f(r, \theta, z)$ in cylindrical coordinates.

$r = 2\sin\theta$

$0 \leq \theta \leq 2\pi$

$0 \leq r \leq 2\sin\theta$





$$\boxed{\begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq r \leq 2\sin\theta \\ 0 \leq z \leq 4-y \end{array}}$$

$$\int_0^{2\pi}\int_0^{2\sin\theta}\int_0^{4-y} f(r,\theta,z)\, r\, dz\, dr\, d\theta$$

$$\iiint_E f(r,\theta,z)\, dV$$

Setting up a Triple Integral in Two Ways

Let E be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. (**Figure 5.53**). Set up a triple integral in cylindrical coordinates to find the volume of the region, using the following orders of integration:

a. $dz\, dr\, d\theta$

b. $dr\, dz\, d\theta.$

cone paraboloid.

$\sqrt{x^2+y^2} \le z \le 2-x^2-y^2$

$r \le z \le 2-r^2$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$V = \iiint_E (\cancel{\quad}) \, dV$

$r\, dz\, dr\, d\theta$

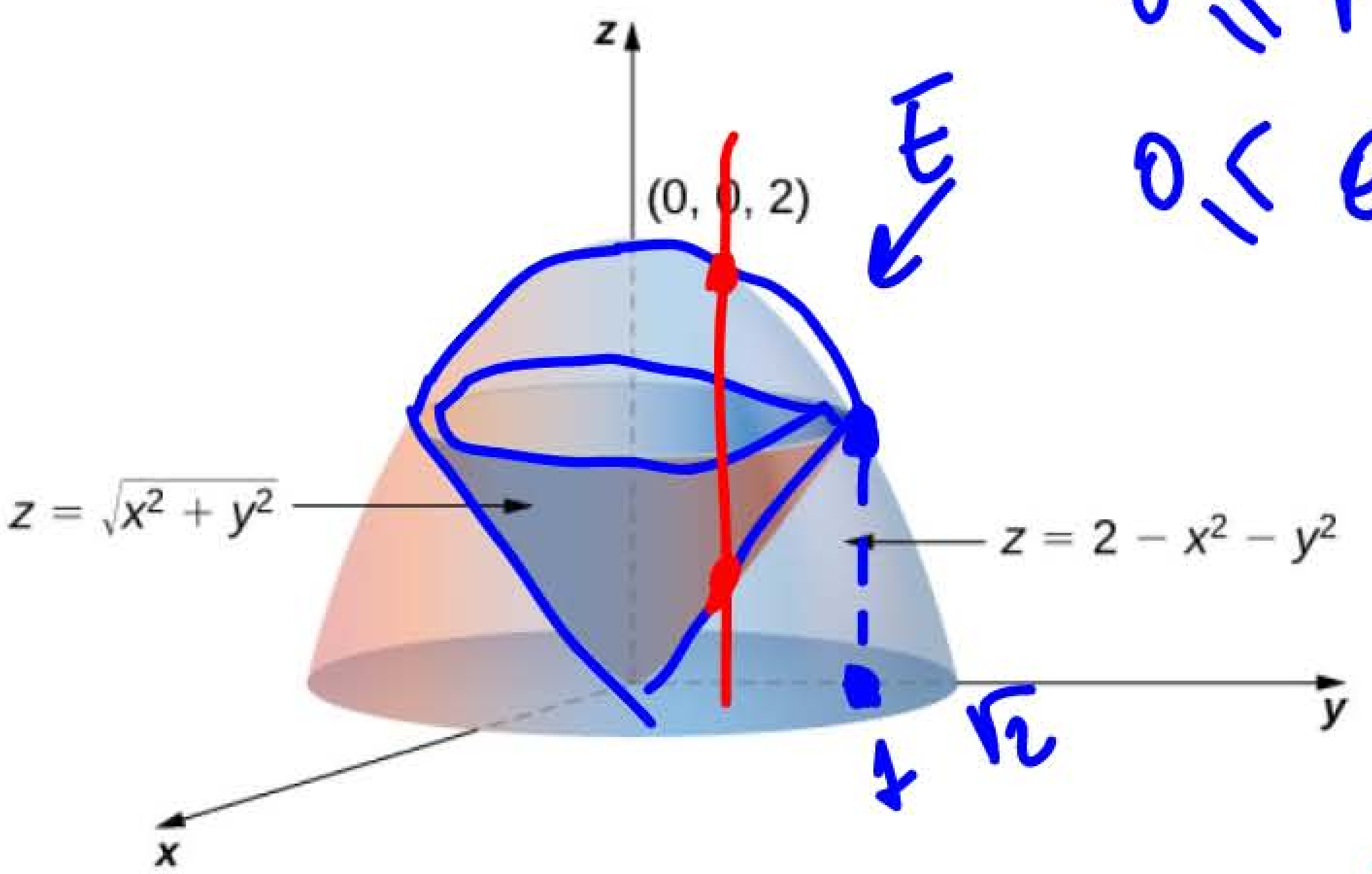
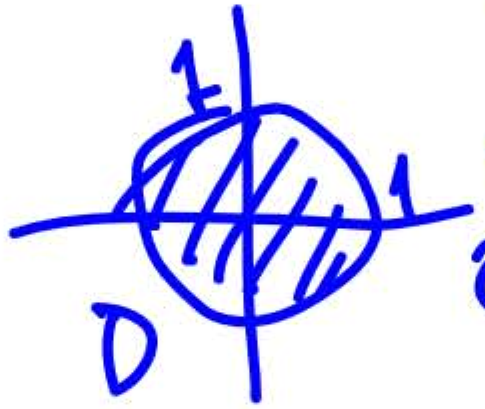


Figure 5.53 Setting up a triple integral in cylindrical coordinates over a conical region.

D: $0 \le r \le 1$

$0 \le \theta \le 2\pi$



$\int_0^{2\pi} \int_0^1 \int_r^{2-r^2} (\cancel{\quad}) \, r \, dz\, dr\, d\theta$

1.

double integral $\int_0^{2\pi} \int_0^1 (2-r^2-r)\, \underbrace{r\, dr\, d\theta}_{dA}$

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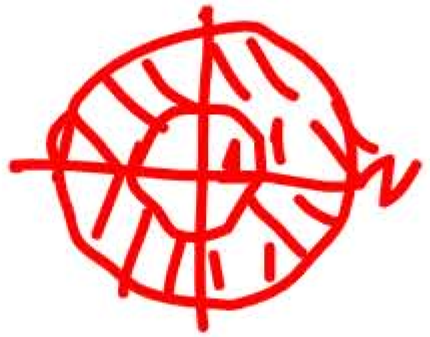
Finding a Volume with Triple Integrals in Two Ways

Let E be the region bounded below by the $r\theta$-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$ (**Figure 5.54**). Set up a triple integral in cylindrical coordinates to find the volume of the region using the following orders of integration, and in each case find the volume and check that the answers are the same:

a. $dz\,dr\,d\theta$

Volume of $E = \iiint_E 1 . dV$



$r^2 + z^2 = 4$

$0 \leq z \leq \sqrt{4-r^2}$

$1 \leq r \leq 2$

$0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi}\int_1^{2}\int_0^{\sqrt{4-r^2}} 1 .\, r\,dz\,dr\,d\theta$$

Volume -

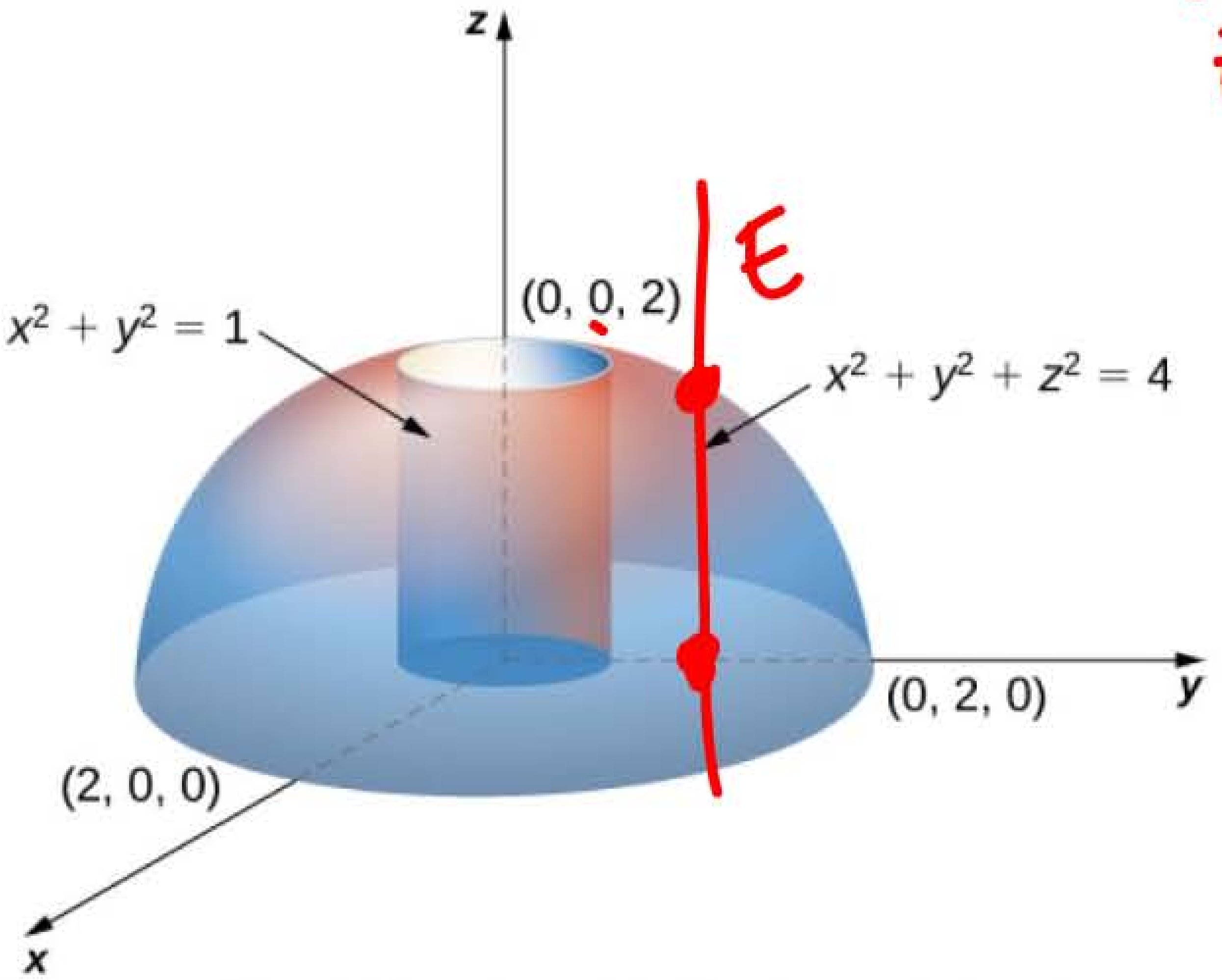


Figure 5.54 Finding a cylindrical volume with a triple integral in cylindrical coordinates.

$$\int_0^{2\pi}\int_1^{2}\int_0^{\sqrt{4-r^2}} 1\cdot r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_1^{2} rz\Big|_0^{\sqrt{4-r^2}}\,dr\,d\theta = \int_0^{2\pi}\boxed{\int_1^{2} r\sqrt{4-r^2}\,dr}\,d\theta$$

$u = 4-r^2$

$du = -2r\,dr$

$-\frac{du}{2} = r\,dr$

$r=2 \Rightarrow u=0$

$r=1 \Rightarrow u=3$

$$\int_3^{0} u^{1/2}\left(-\frac{du}{2}\right) = -\frac{1}{2}\,\frac{u^{3/2}}{3/2}\Big|_3^{0} = -\frac{1}{3}\left(0-3\sqrt{3}\right)$$

$$=\sqrt{3}$$

$$\int_0^{2\pi}\sqrt{3}\,d\theta = \sqrt{3}\,\theta\Big|_0^{2\pi} = 2\sqrt{3}\,\pi \text{ unit}^3.$$

Spherical Coordinates

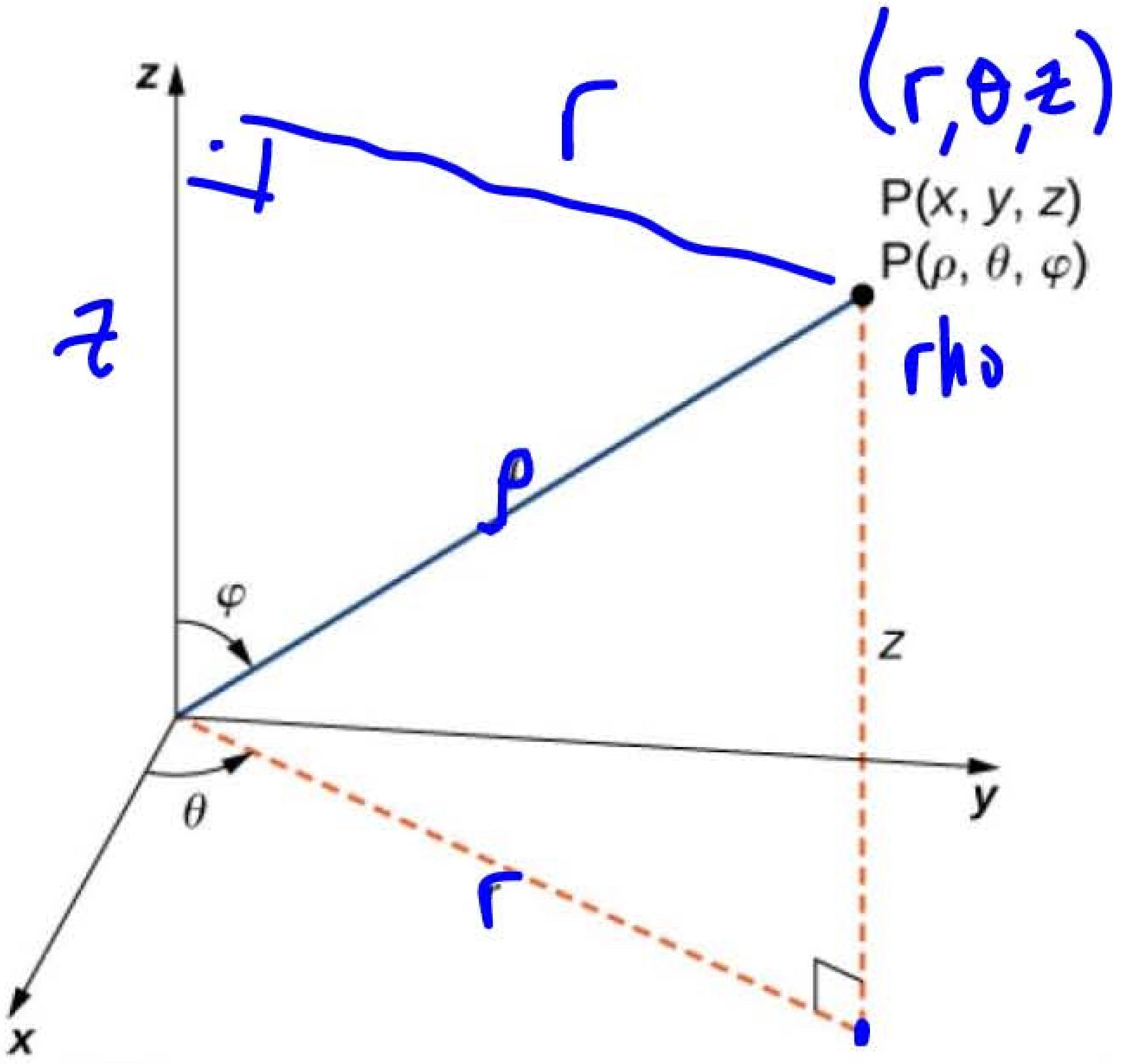


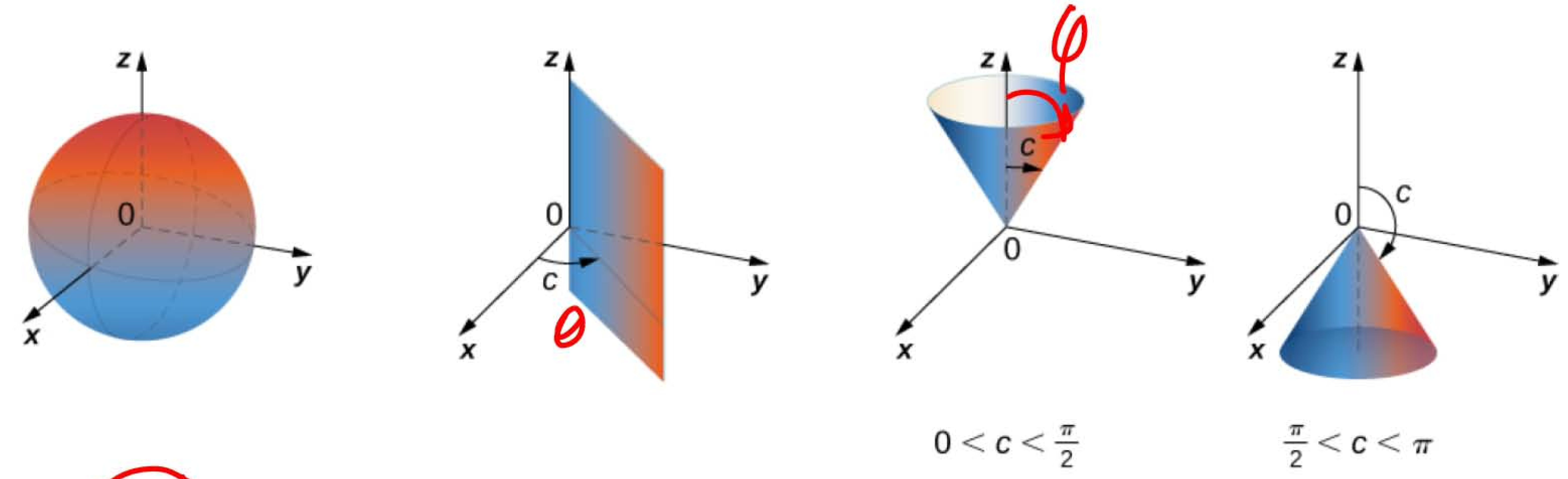
Figure 5.55 The spherical coordinate system locates points with two angles and a distance from the origin.

$$r = \rho \sin \varphi$$

$$x = \rho \sin \varphi \cos \theta$$

$$y = \rho \sin \varphi \sin \theta$$

$$z = \rho \cos \varphi$$



Sphere $\rho = c$ (constant) Half plane $\theta = c$ (constant) Half cone $\varphi = c$ (constant)

Figure 5.56 Spherical coordinates are especially convenient for working with solids bounded by these types of surfaces. (The letter c indicates a constant.)

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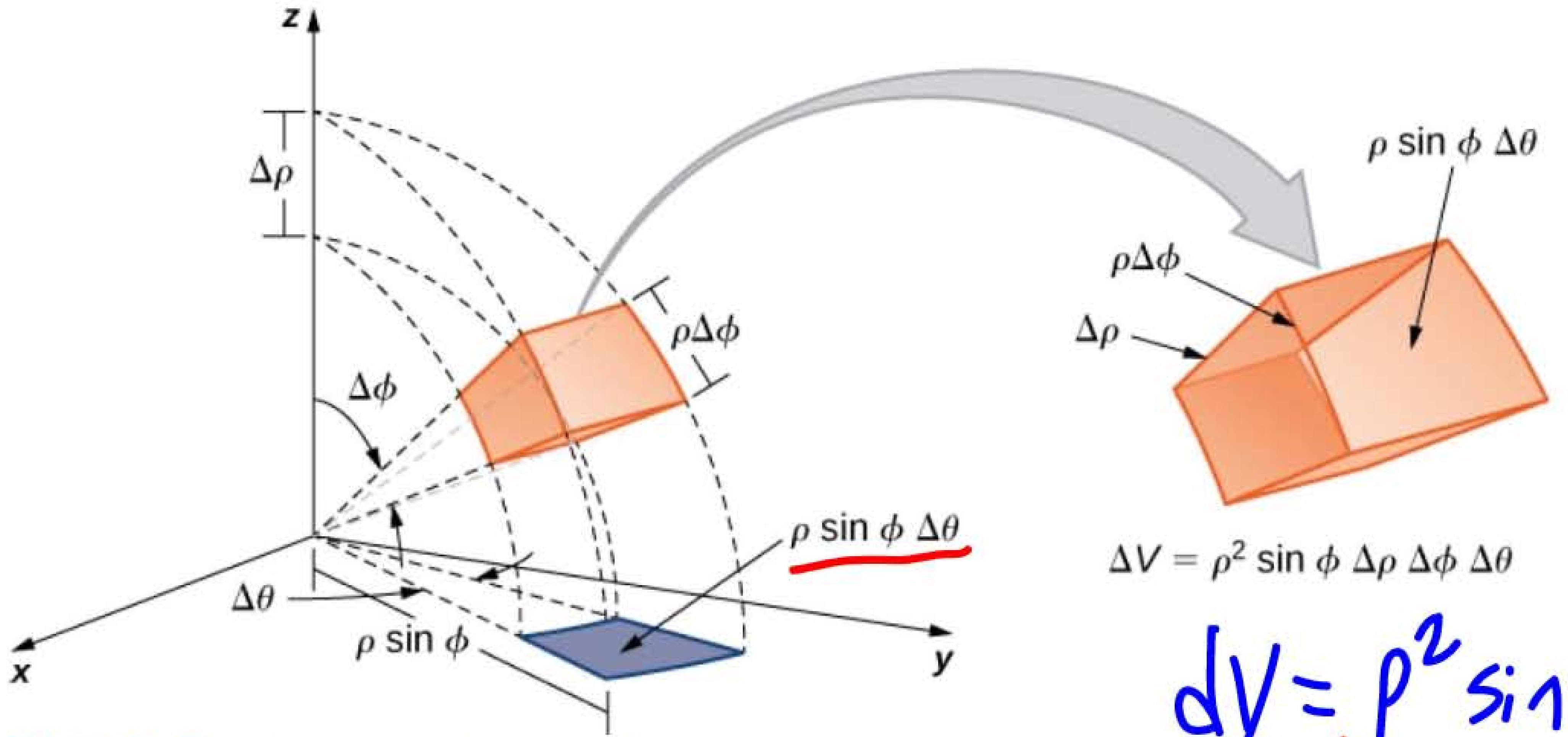


$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

Figure 5.57 The volume element of a box in spherical coordinates.

$$= r \, dr \, d\theta \, dz$$

Theorem 5.13: Fubini's Theorem for Spherical Coordinates

$$dV = dx\,dy\,dz$$

If $f(\rho, \theta, \varphi)$ is continuous on a spherical solid box $B = [a, b] \times [\alpha, \beta] \times [\gamma, \psi]$, then

$$\iiint_B f(\rho, \theta, \varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \int_{\varphi=\gamma}^{\varphi=\psi} \int_{\theta=\alpha}^{\theta=\beta} \int_{\rho=a}^{\rho=b} f(\rho, \theta, \varphi)\boxed{\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta}. \quad (5.12)$$

This iterated integral may be replaced by other iterated integrals by integrating with respect to the three variables in other orders.

Evaluating a Triple Integral in Spherical Coordinates

Evaluate the iterated triple integral $\displaystyle\int_{\theta=0}^{\theta=2\pi}\int_{\varphi=0}^{\varphi=\pi/2}\int_{\rho=0}^{\rho=1} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$

box $0 \le \theta \le 2\pi$

$0 \le \varphi \le \pi/2$

$0 \le \rho \le 1$

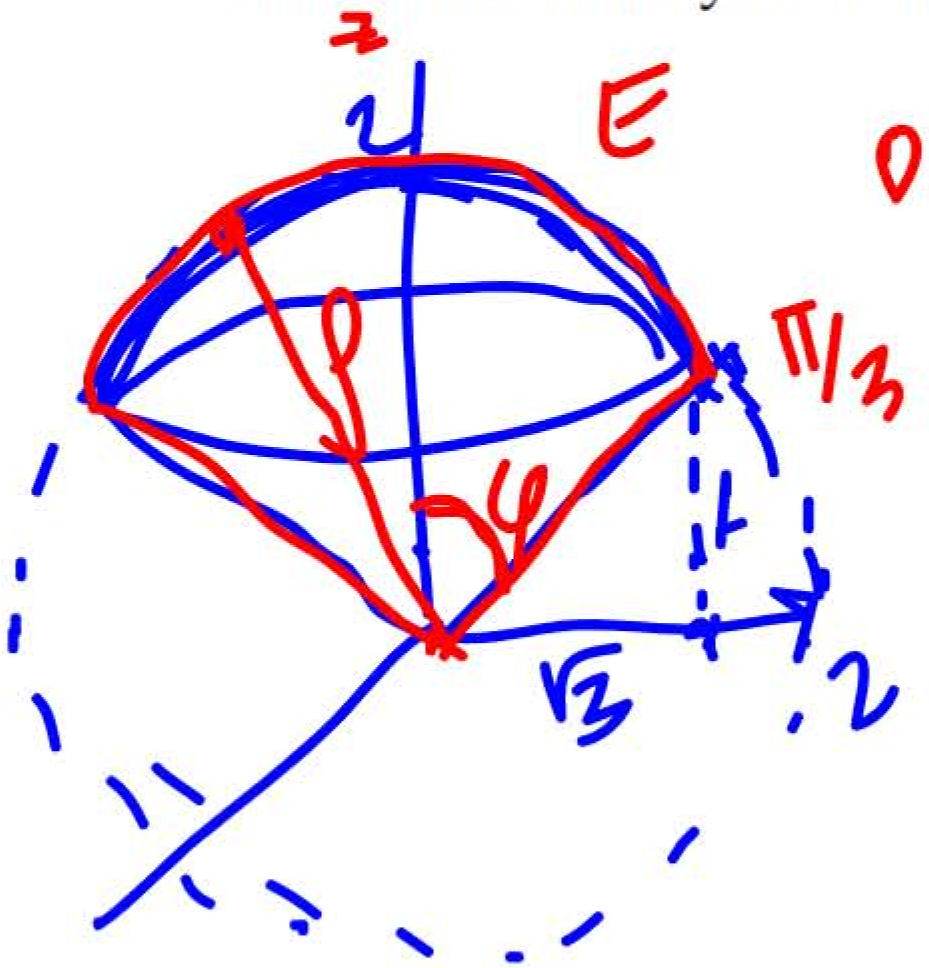
rectangular box

$$\int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^3}{3}\sin\varphi \Big|_0^1 \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi/2} \frac{\sin\varphi}{3} \, d\varphi \, d\theta$$

$$\int_0^{2\pi} -\frac{\cos\varphi}{3}\Big|_0^{\pi/2} d\theta = \int_0^{2\pi} \frac{1}{3} \, d\theta = \frac{\theta}{3}\Big|_0^{2\pi}$$

$$= \frac{2\pi}{3}$$

5.31 Set up a triple integral for the volume of the solid region bounded above by the sphere $\rho = 2$ and bounded below by the cone $\varphi = \pi/3$.



$0 \leq \rho \leq 2$

$0 \leq \varphi \leq \frac{\pi}{3}$

$0 \leq \theta \leq 2\pi$

$$V = \iiint_E 1 \, dV = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{2} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

spherical coordinates

cylindrical coordinates $\frac{r}{\sqrt{3}} \leq z \leq \sqrt{4-r^2}$

$0 \leq r \leq \sqrt{3}$

$0 \leq \theta \leq 2\pi$

$x^2+y^2+z^2 = 2^2$

$z = \sqrt{4-r^2}$

$z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$

$$V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r/\sqrt{3}}^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta$$

Converting from Rectangular Coordinates to Cylindrical Coordinates

x, y, z $\qquad\qquad$ r, θ, z

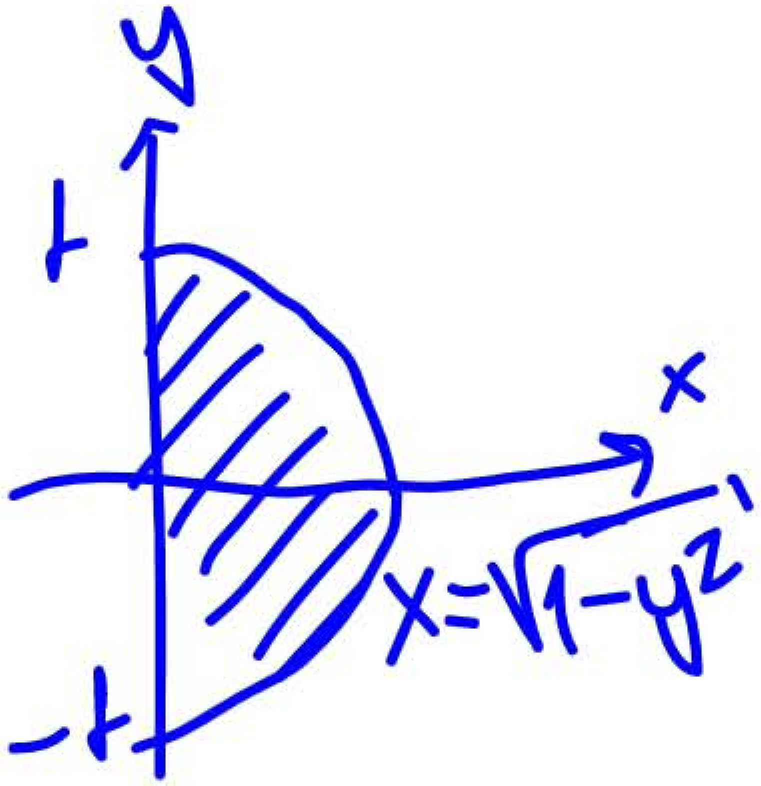
Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz\, \underbrace{dz\, dx\, dy}_{}.$$

dV

$dV = dx\,dy\,dz$

$= r\,dr\,d\theta\,dz$



$-1 \le y \le 1$

$0 \le x \le \sqrt{1-y^2}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 1$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1} \int_{r^2}^{r} \underbrace{r\cos\theta}\, \underbrace{r\sin\theta}\, z\, r\, dz\, dr\, d\theta$$

$\theta \quad r \quad z$

$x^2+y^2 = r^2$

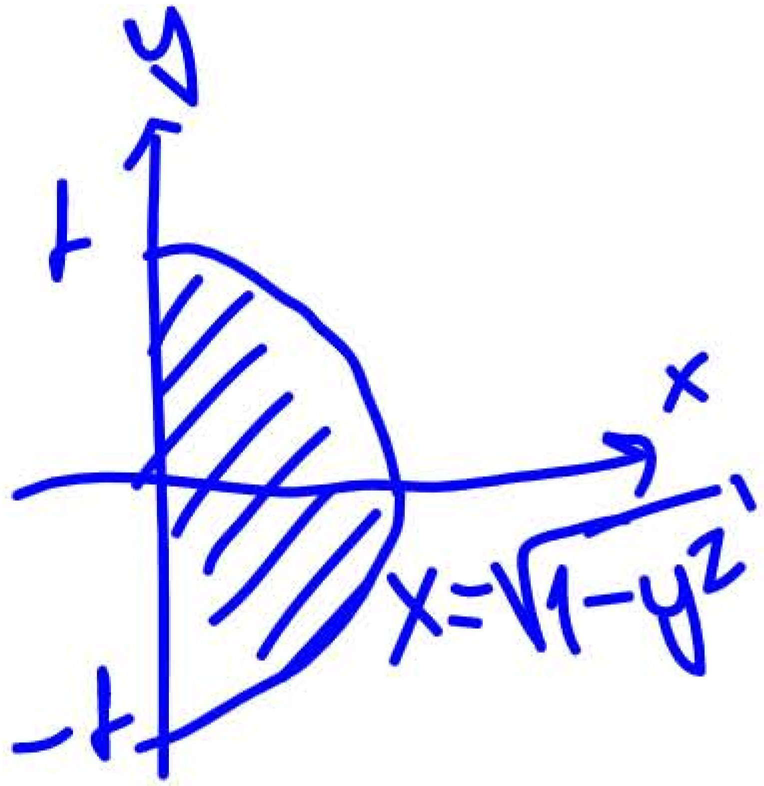
Converting from Rectangular Coordinates to Cylindrical Coordinates

x, y, z $\qquad\qquad$ r, θ, z

Convert the following integral into cylindrical coordinates:

$$\int_{y=-1}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} \int_{z=x^2+y^2}^{z=\sqrt{x^2+y^2}} xyz \, \underbrace{dz\, dx\, dy}_{dV}.$$

$$dV = dx\,dy\,dz$$
$$= r\,dr\,d\theta\,dz$$



$-1 \le y \le 1$

$0 \le x \le \sqrt{1-y^2}$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 1$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{1} \int_{r^2}^{r} r\cos\theta \; r\sin\theta \; z \; r \, dz\, dr\, d\theta$$

$\theta \qquad r \qquad z$

$x^2+y^2=r^2$

Converting from Rectangular Coordinates to Spherical Coordinates

Convert the following integral into spherical coordinates:

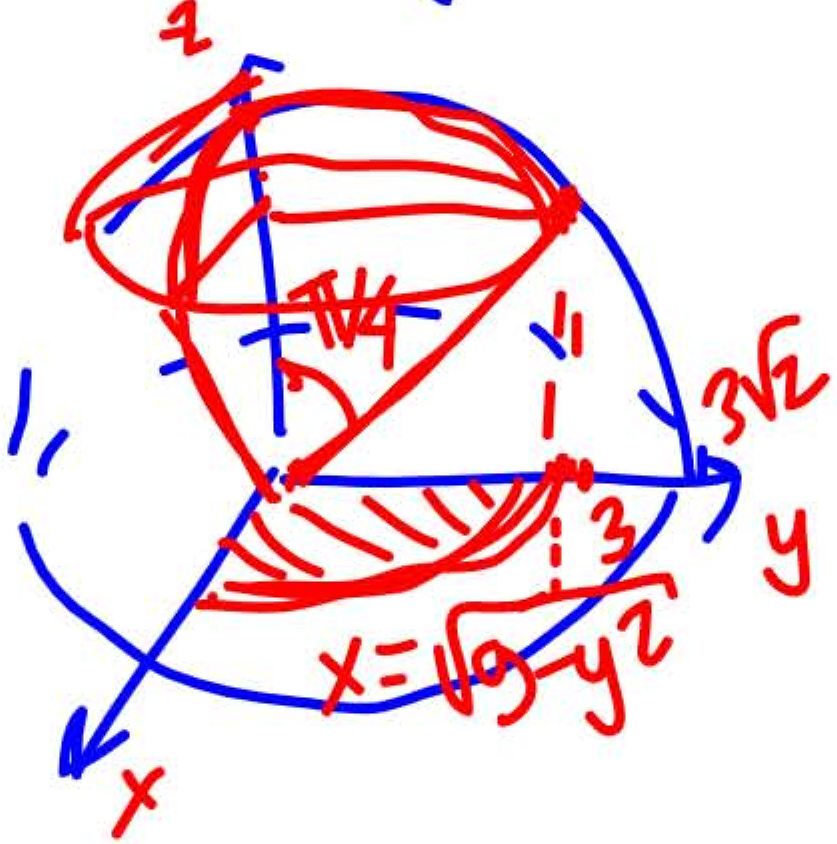
$$\int_{y=0}^{y=3}\int_{x=0}^{x=\sqrt{9-y^2}}\int_{z=\sqrt{x^2+y^2}}^{z=\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)\,dz\,dx\,dy.$$

(ρ, θ, φ) $\quad x^2+y^2+z^2=\rho^2$

$dV = \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$

$z^2+x^2+y^2=18=(3\sqrt{2})^2$ sphere

$z=\sqrt{x^2+y^2}$ cone



$0 \leq \rho \leq 3\sqrt{2}$

$0 \leq \varphi \leq \pi/4$

$0 \leq \theta \leq \pi/2$

$0 \leq r \leq 3$

y

$$\int_0^{\pi/2}\int_0^{\pi/4}\int_0^{3\sqrt{2}} \rho^2 \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

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100%