6 | VECTOR CALCULUS

Chapter Outline

6.1 Vector Fields
6.2 Line Integrals ✓
6.3 Conservative Vector Fields ✓
6.4 Green's Theorem
6.5 Divergence and Curl
6.6 Surface Integrals
6.7 Stokes' Theorem
6.8 The Divergence Theorem

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56. Evaluate $\int_\gamma (x^2+y^2+z^2)^{-1}\,ds$, where γ is the helix $x=\cos t,\ y=\sin t,\ z=t\,(0\le t\le T)$.

$$ds = \sqrt{x'(t)^2 + y'(t)^2 + (z'(t))^2}\,dt$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1^2}\,dt = \sqrt{2}\,dt$$

$$\int_0^T \frac{1}{\cos^2 t + \sin^2 t + t^2}\sqrt{2}\,dt$$

$$= \int_0^T \frac{1}{1+t^2}\sqrt{2}\,dt = \arctan t\,\Big|_0^T \sqrt{2}.$$

$\tan 0 = 0$

$\arctan 0 = 0$

$$= \sqrt{2}\left[\arctan T - \arctan 0\right]$$

$$= \sqrt{2}\arctan T$$

57. Evaluate $\int_C yz\,dx + xz\,dy + xy\,dz$ over the line segment from $(1, 1, 1)$ to $(3, 2, 0)$.

A B

$\vec{u} = \langle 3-1, 2-1, 0-1 \rangle$ direction vector $\overrightarrow{AB}$

$\ell: \langle 1,1,1 \rangle + t\langle 2,1,-1 \rangle$

$= \langle 1+2t, 1+t, 1-t \rangle$

$x(t) = 1+2t \qquad dx = 2dt$

$y(t) = 1+t \qquad dy = dt$

$z(t) = 1-t \qquad dz = -dt$

from $t=0$ (beginning, A) to $t=1$ (B)

$\int_C F \cdot dr \qquad \int_C \langle M,N \rangle \cdot \langle dx,dy \rangle$

$\int_C \langle P,Q,R \rangle \cdot \langle dx,dy,dz \rangle$

$$\int_0^1 \left[(1+t)(1-t)\,2dt + (1+2t)(1-t)\,dt + (1+2t)(1+t)(-dt)\right]$$

$$= \int_0^1 \left[2-2t^2+1+t-2t^2 \cancel{+} -1-3t-2t^2\right]dt$$

$$= \int_0^1 (-6t^2-2t+2)\,dt = -1$$

$$= \left(-2t^3 - t^2 + 2t\right)\Big|_0^1 = -2 - 1 + 2 = \underline{-1}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

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Theorem 6.5: Properties of Vector Line Integrals

Let $\mathbf{F}$ and $\mathbf{G}$ be continuous vector fields with domains that include the oriented smooth curve C. Then

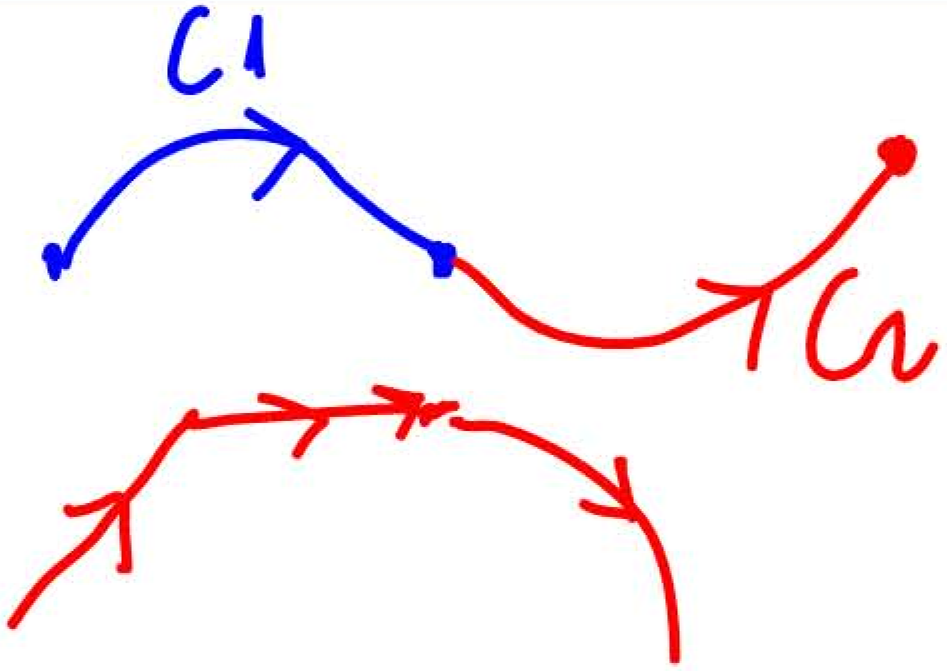
i. $\int_C (\mathbf{F}+\mathbf{G}) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{G} \cdot d\mathbf{r}$

ii. $\int_C k\mathbf{F} \cdot d\mathbf{r} = k \int_C \mathbf{F} \cdot d\mathbf{r}$, where k is a constant

iii. $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}$

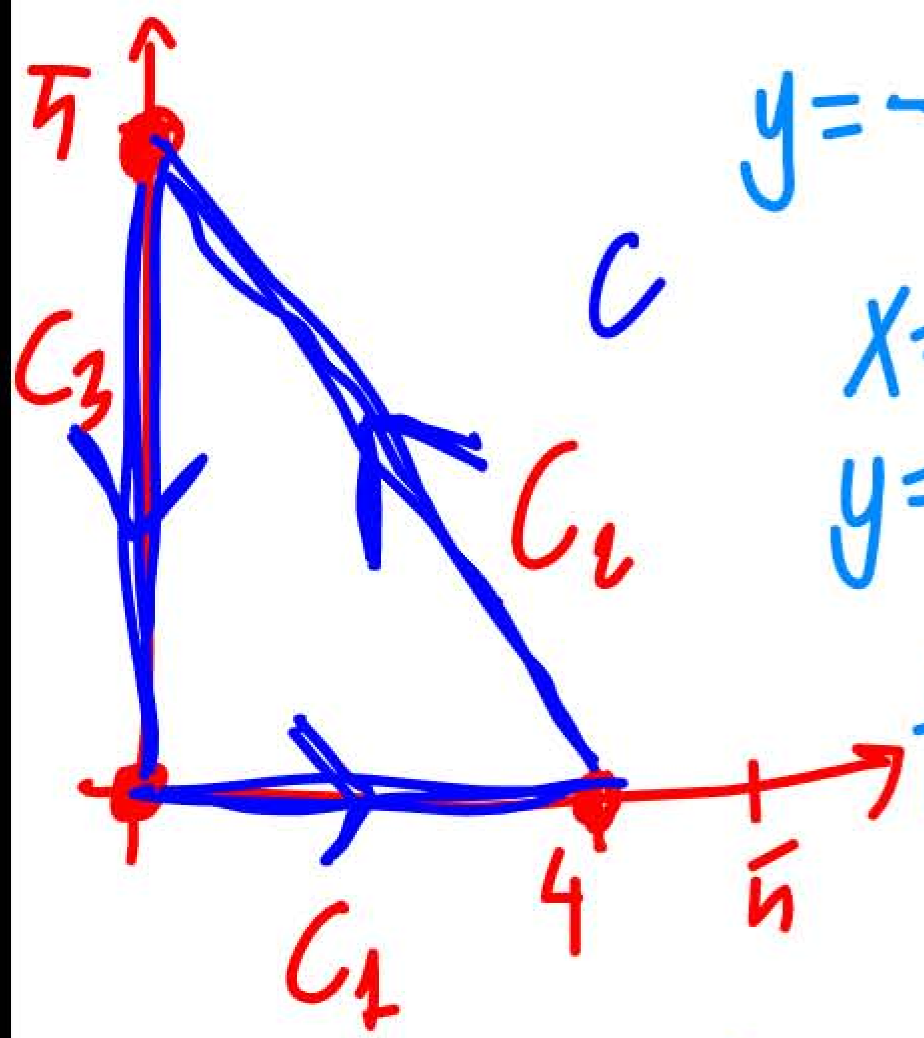
iv. Suppose instead that C is a piecewise smooth curve in the domains of $\mathbf{F}$ and $\mathbf{G}$, where $C = C_1 + C_2 + \cdots + C_n$ and $C_1, C_2, \ldots, C_n$ are smooth curves such that the endpoint of C_i is the starting point of C_{i+1}. Then

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} + \int_{C_2} \mathbf{F} \cdot d\mathbf{s} + \cdots + \int_{C_n} \mathbf{F} \cdot d\mathbf{s}.$$



$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

6.19 Calculate line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}$ is vector field $\langle y^2, 2xy+1\rangle$ and C is a triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 5)$, oriented counterclockwise.



C

$y=-\frac{5x}{4}+5$

$x=t$

$y=-\frac{5t}{4}+5$

from $t=4$ to $t=0$

$y=0$

$x=t$

$0\leq t\leq 4$

$C_3: x=0$

$y=t$

t is from 5 to 0.

$$\int_0^4 \langle 0,1\rangle\cdot\langle dt,0\rangle = \int_0^4 0\,dt = 0 = \int_{C_1} F\cdot dr$$

$$\int_4^0 \left\langle \left(-\frac{5t}{4}+5\right)^2, 2t\left(-\frac{5t}{4}+5\right)+1\right\rangle\cdot\left\langle dt, -\frac{5dt}{4}\right\rangle = \int_{C_2} F\cdot dr$$

$x=0,\ dx=0.$

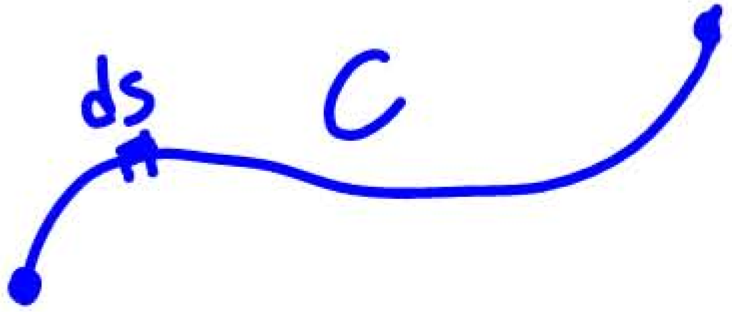
$y=t\ \ dy=dt$

$$\int_{C_3} F\cdot dr = \int_5^0 \langle t^2,1\rangle\cdot\langle 0,dt\rangle = \int_5^0 dt = t\Big|_5^0 = -5$$

Applications of Line Integrals

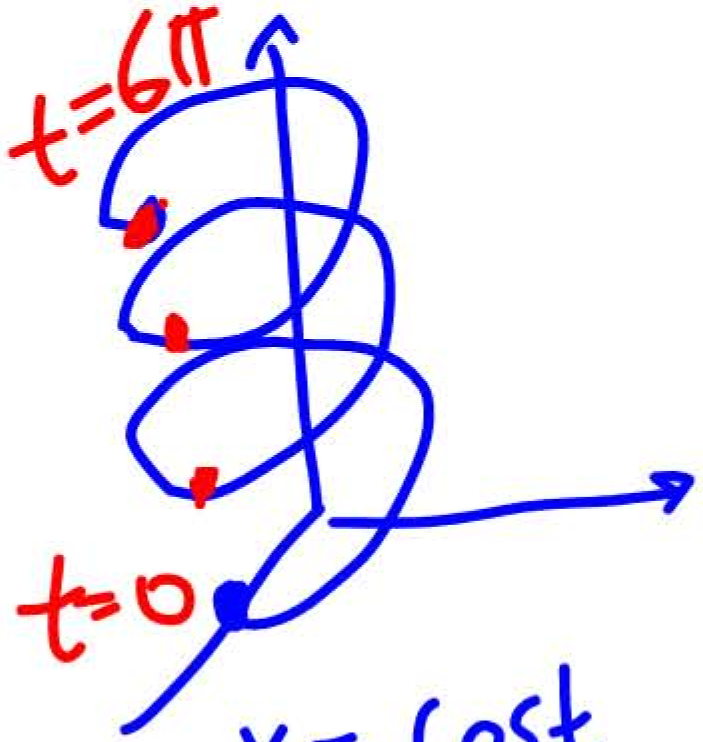
Calculating the Mass of a Wire

$$M = \int_C \rho(x, y, z)\,ds.$$



$$dm = \rho \cdot ds$$

6.20 Calculate the mass of a spring in the shape of a helix parameterized by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 6\pi$, with a density function given by $\rho(x, y, z) = x + y + z$ kg/m.



$x = \cos t$

$y(t) = \sin t$

$z = t$

$$ds = \sqrt{\sin^2 t + \cos^2 t + 1^2}\,dt$$

$$\int_0^{6\pi} (\cos t + \sin t + t)\sqrt{2}\,dt$$

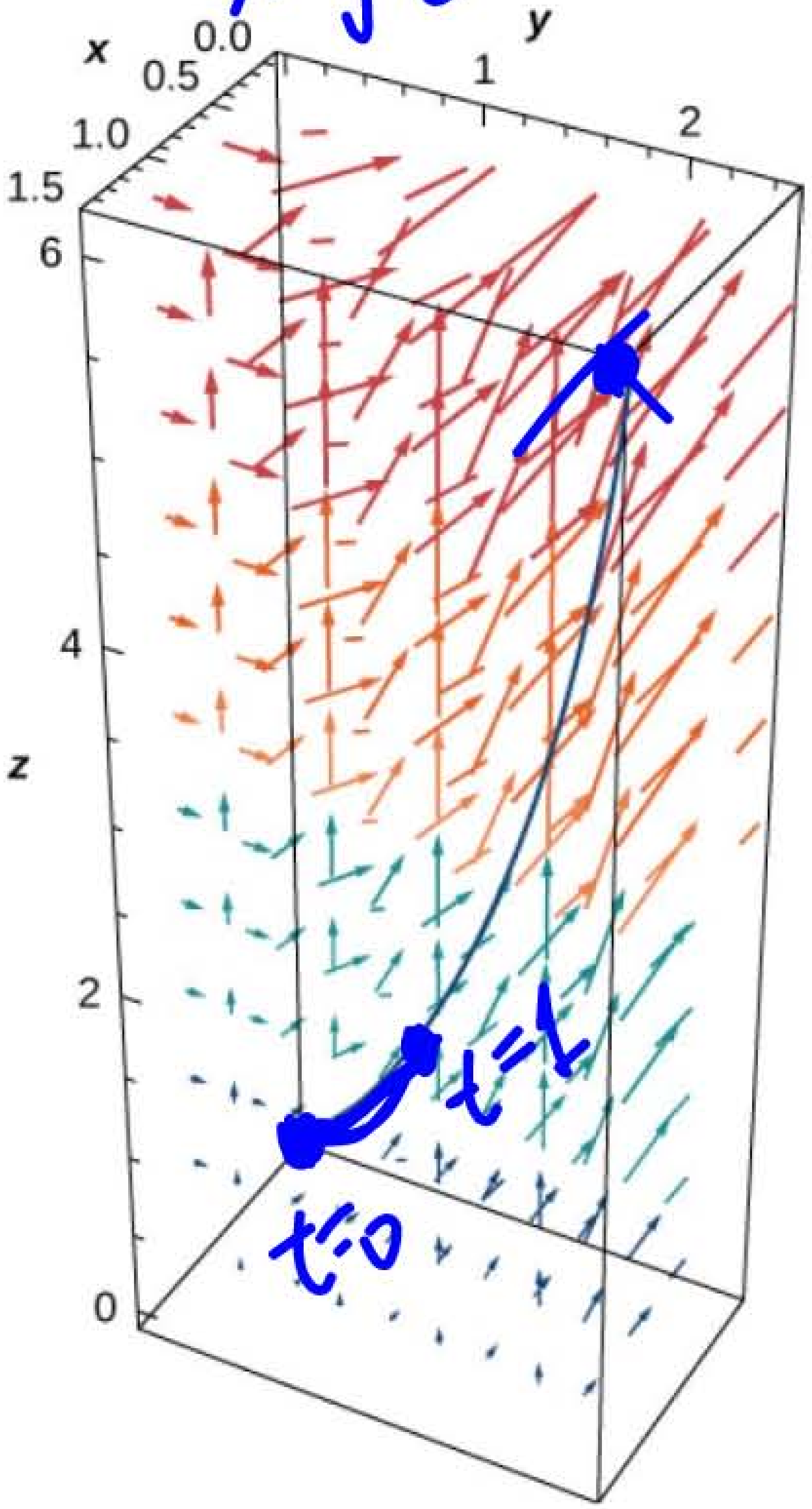
$$= \sqrt{2}\left[\sin t - \cos t + \frac{t^2}{2}\right]_0^{6\pi}$$

$$= \sqrt{2}\left[0 - 1 + 18\pi^2 - (0 - 1 + 0)\right] = 18\sqrt{2}\,\pi^2 \text{ kg}$$

Calculating Work

How much work is required to move an object in vector force field $\mathbf{F} = \langle yz, xy, xz \rangle$ along path $\mathbf{r}(t) = \langle t^2, t, t^4 \rangle$, $0 \le t \le 1$? See **Figure 6.22**.

x y z



Work = Force × distance

$F \cdot dr$

(0,0,0)

$$\int_0^1 \langle t^5, t^3, t^6 \rangle \cdot \langle 2t, 1, 4t^3 \rangle \, dt$$

$x = t^2 \quad dx = 2t\,dt$

$y = t \quad dy = dt$

$z = t^4 \quad dz = 4t^3\,dt$

$$\int_0^1 (2t^6 + t^3 + 4t^9)\,dt$$

$$\left(\frac{2t^7}{7} + \frac{t^4}{4} + \frac{2t^{10}}{5}\right)\Bigg|_0^1$$

$$= \frac{2}{7} + \frac{1}{4} + \frac{2}{5} = \frac{40+35+56}{140}$$

$$\frac{131}{140}$$

Flux and Circulation

Flux is used in applications to calculate fluid flow across a curve $\int_C \mathbf{F} \cdot \mathbf{N} ds$

$\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$ $\mathbf{N}(t) = \dfrac{\mathbf{n}(t)}{\| \mathbf{n}(t) \|}$ is the unit normal vector to C

tangent $\langle x'(t), y'(t) \rangle$

normal $\langle y'(t), -x'(t) \rangle$

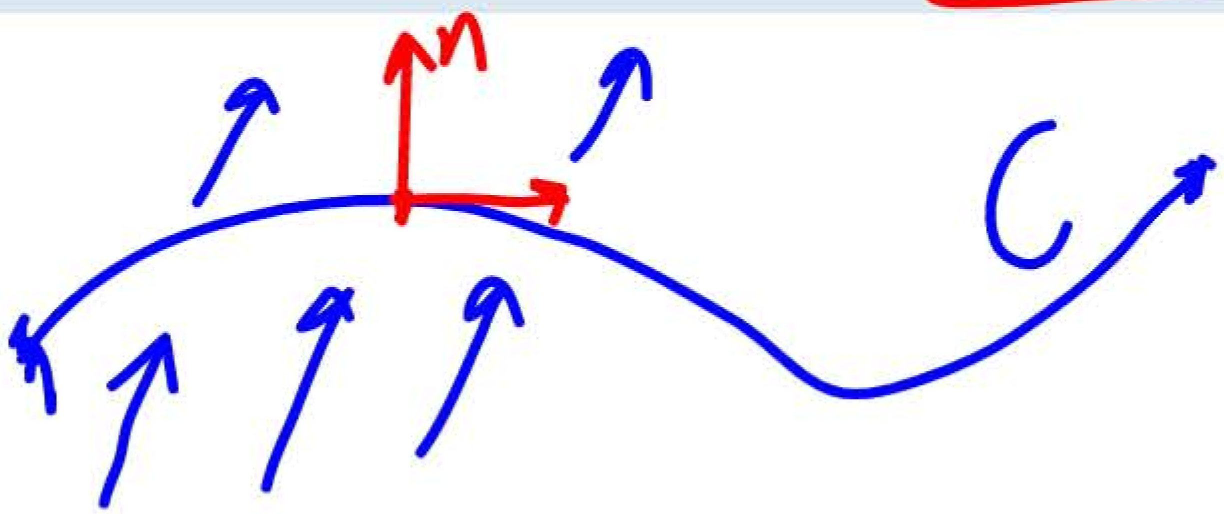
Definition

The **flux** of **F** across C is line integral $\int_C \mathbf{F} \cdot \dfrac{\mathbf{n}(t)}{\| \mathbf{n}(t) \|} ds$.

Theorem 6.6: Calculating Flux across a Curve

Let **F** be a vector field and let C be a smooth curve with parameterization $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$. Let $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$. The flux of **F** across C is

$$\int_C \mathbf{F} \cdot \mathbf{N} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{n}(t) dt \qquad (6.11)$$



$$\frac{n(t)}{\cancel{\sqrt{(y'(t))^2 + (x'(t))^2}}} \cdot \cancel{\sqrt{(x'(t))^2 + (y'(t))^2}}\, dt$$

6.21 Calculate the flux of $\mathbf{F} = \langle x+y, 2y \rangle$ across the line segment from $(0, 0)$ to $(2, 3)$, where the curve is oriented from left to right.

$\vec{u} = \langle 2, 3 \rangle$

$l: \langle 0, 0 \rangle + t\langle 2, 3 \rangle$

$x = 2t$

$y = 3t$

$0 \leq t \leq 1$

$r(t) = \langle 2t, 3t \rangle$

$n(t) = \langle 3, -2 \rangle$

$$\int_0^1 F(r(t)) \cdot n(t)\,dt$$

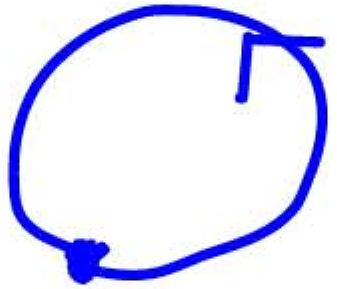
$$\int_0^1 \langle 2t+3t, 6t \rangle \cdot \langle 3, -2 \rangle\,dt$$

$$\int_0^1 3t\,dt = 3\frac{t^2}{2}\Big|_0^1 = \underline{\frac{3}{2}}$$

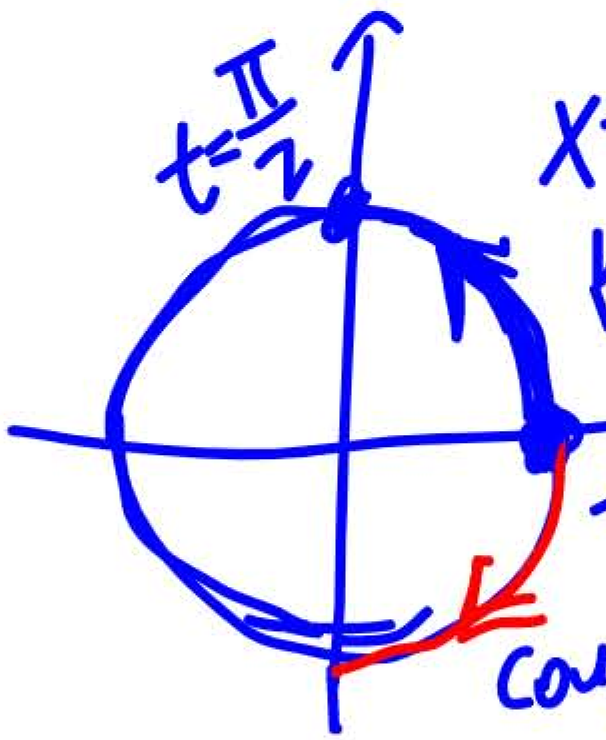
The line integral of vector field **F** along an oriented closed curve is called the **circulation** of **F** along C.

closed

Circulation line integrals have their own notation: $\oint_C \mathbf{F} \cdot \mathbf{T}\,ds$.



Let $\mathbf{F} = \langle -y, x \rangle$ be the vector field from **Example 6.16** and let C represent the unit circle oriented counterclockwise. Calculate the circulation of **F** along C.



$x = \cos t$

$y = \sin t$

$t = 0$ to 2π

counter clockwise

$\langle \cos t, -\sin t \rangle$ clockwise

$$\int_0^{2\pi} \langle -\sin t, \cos t \rangle \cdot \langle \cos t, \sin t \rangle\, dt$$

$r(t) = \langle \cos t, \sin t \rangle$

$r'(t) = \langle -\sin t, \cos t \rangle$

$r(t) = \langle \cos t, \sin t \rangle$

$$= \int_0^{2\pi} 0\,dt$$

$$= 0$$

$$\oint_C F \cdot T\,ds = 0$$

6.3 | Conservative Vector Fields

If $F = \nabla f$ then F is conservative

Definition

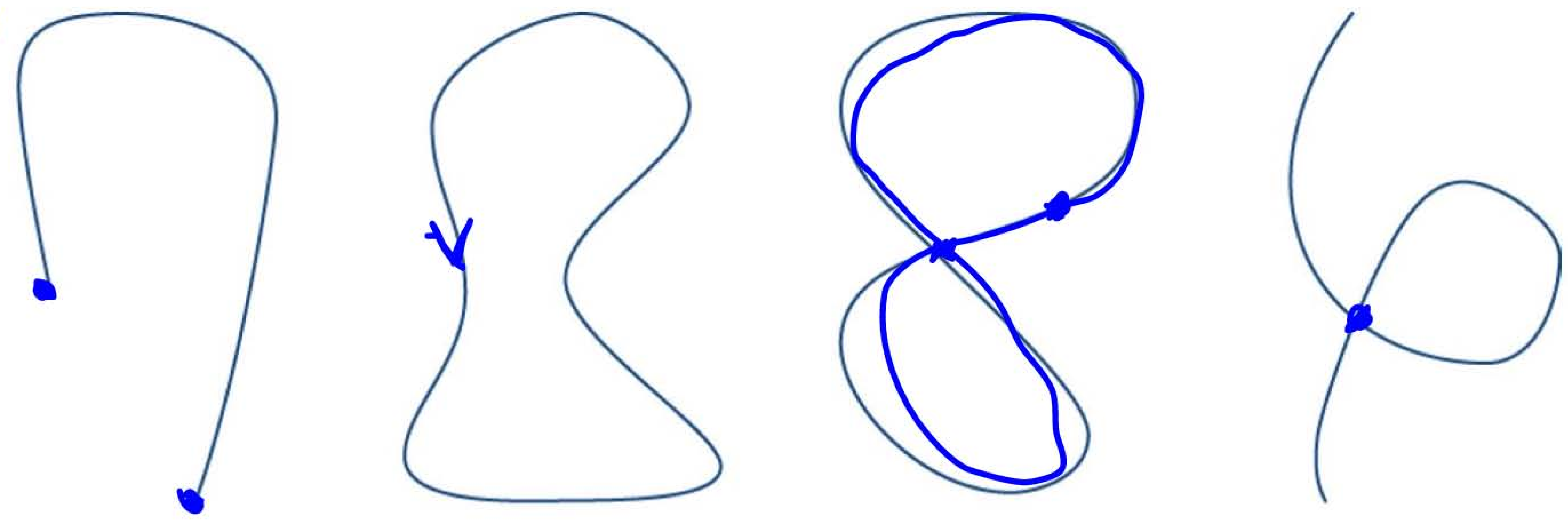
Curve C is a **closed curve** if there is a parameterization $\mathbf{r}(t),\ a \le t \le b$ of C such that the parameterization traverses the curve exactly once and $\mathbf{r}(a) = \mathbf{r}(b)$. Curve C is a **simple curve** if C does not cross itself. That is, C is simple if there exists a parameterization $\mathbf{r}(t),\ a \le t \le b$ of C such that $\mathbf{r}$ is one-to-one over (a, b). It is possible for $\mathbf{r}(a) = \mathbf{r}(b)$, meaning that the simple curve is also closed.

$F = \langle P, Q \rangle$

$P_y = Q_x$

f

potential function



(a) Simple, not closed

(b) Simple, closed

(c) Not simple, closed

(d) Not simple, not closed

6.24 Is the curve given by parameterization $\mathbf{r}(t) = \langle\, 2\cos t,\ 3\sin t\,\rangle,\ 0 \le t \le 2\pi$ a simple closed curve?

ellipse

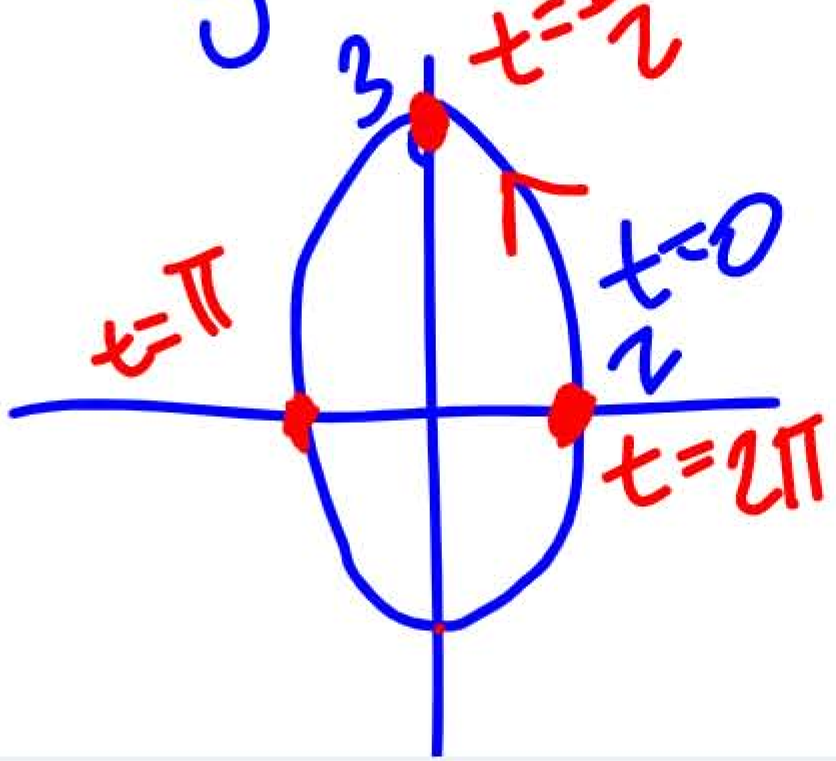
YES, it is simple and closed.

$x = 2\cos t$

$y = 3\sin t$

$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$

$\frac{x^2}{4} + \frac{y^2}{9} = 1$



$0 \le t \le 4\pi$, closed but not simple

$0 \le t \le 3\pi$ not closed not simple.

$r(0) = (2,0)$

$r(3\pi) = (-2,0)$

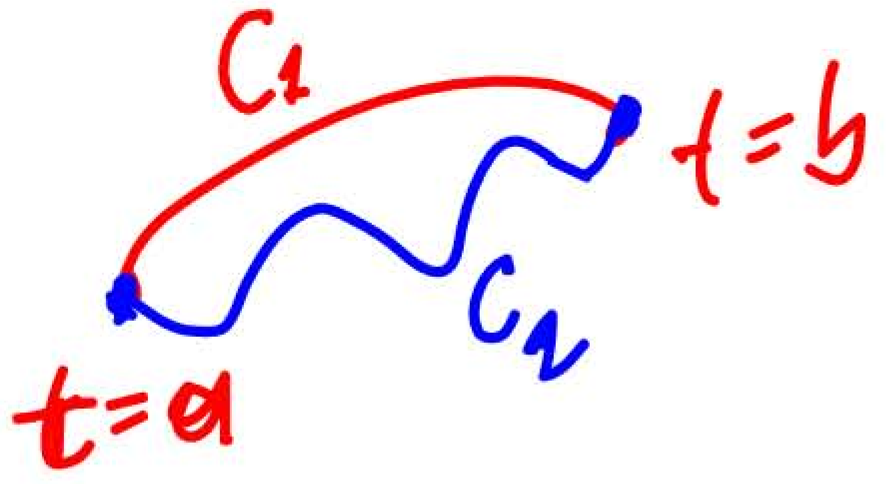
Theorem 6.7: The Fundamen

Let C be a piecewise smooth curve with parameterization $\mathbf{r}(t)$, $a \le t \le b$. Let f be a function of two or three variables with first-order partial derivatives that exist and are continuous on C. Then,

$$\int_C \nabla f \bullet d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{6.12}$$

∇f gradient.

Observe that RHS (right hand side) is independent of path.

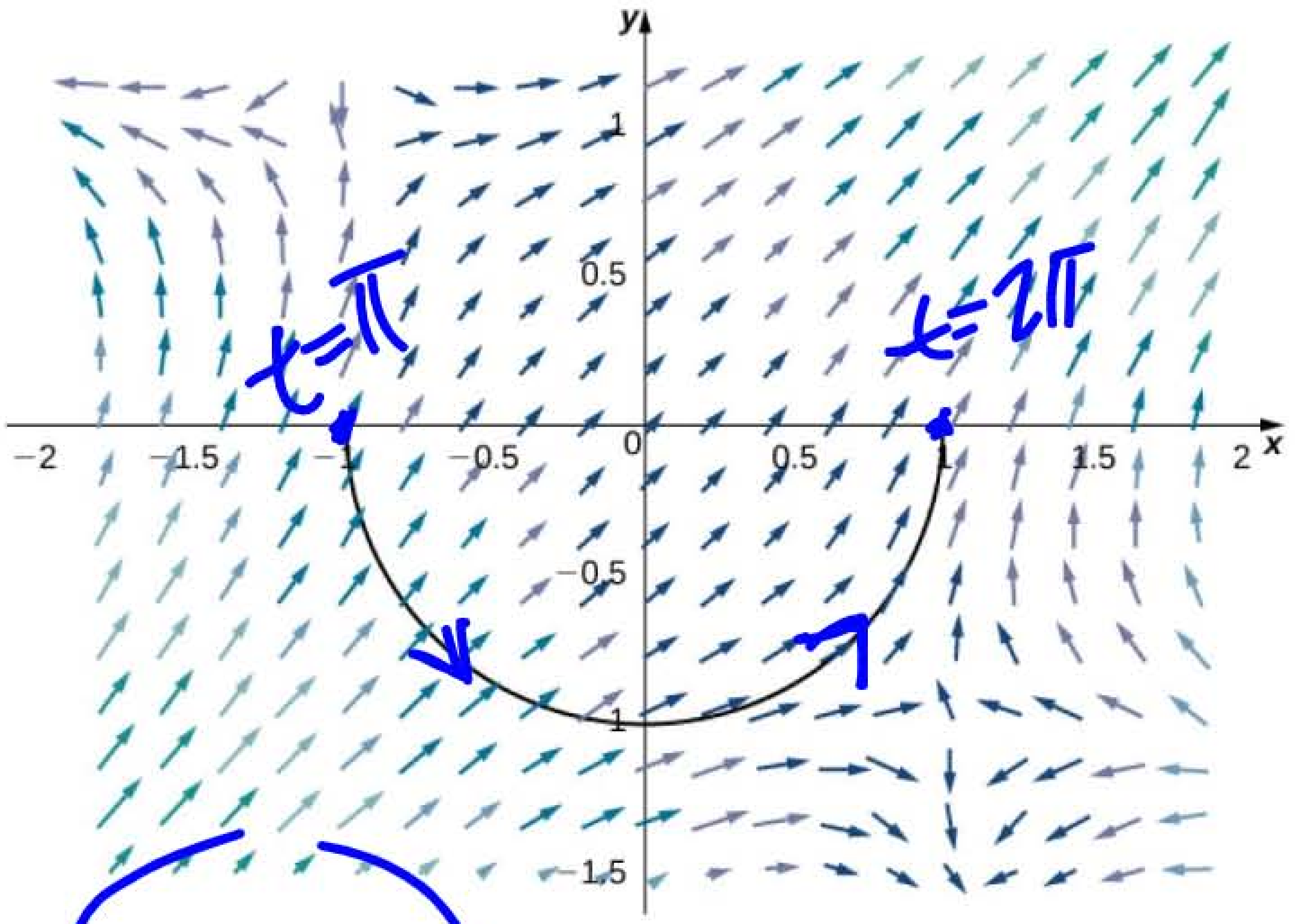


Calculus 1

$f'(x) = F(x)$

$$\int_a^b F(x)\,dx = f(b) - f(a)$$

6.26 Given that $f(x, y) = (x-1)^2 y + (y+1)^2 x$ is a potential function for $\mathbf{F} = \langle 2xy - 2y + (y+1)^2, (x-1)^2 + 2yx + 2x \rangle$, calculate integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the lower half of the unit circle oriented counterclockwise.



$$\nabla f = F = \langle f_x, f_y \rangle$$

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(r(b)) - f(r(a))$$

$$= f(1,0) - f(-1,0)$$

$$= 0 + 1 - ((-2)^2 \cdot 0 + 1^2 \cdot (-1))$$

$$= \underline{2}$$

$x = \cos t$
$y = \sin t$

$\pi \leq t < 2\pi$

$$r(t) = \langle \underbrace{\cos t}_{x}, \underbrace{\sin t}_{y} \rangle$$

Definition

Let **F** be a vector field with domain D. The vector field **F** is **independent of path** (or **path independent**) if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any paths C_1 and C_2 in D with the same initial and terminal points.

Theorem 6.8: Path Independence of Conservative Fields

If **F** is a conservative vector field, then **F** is independent of path.

$$F = \langle \overset{f_x}{y}, \overset{f_y}{x} \rangle$$

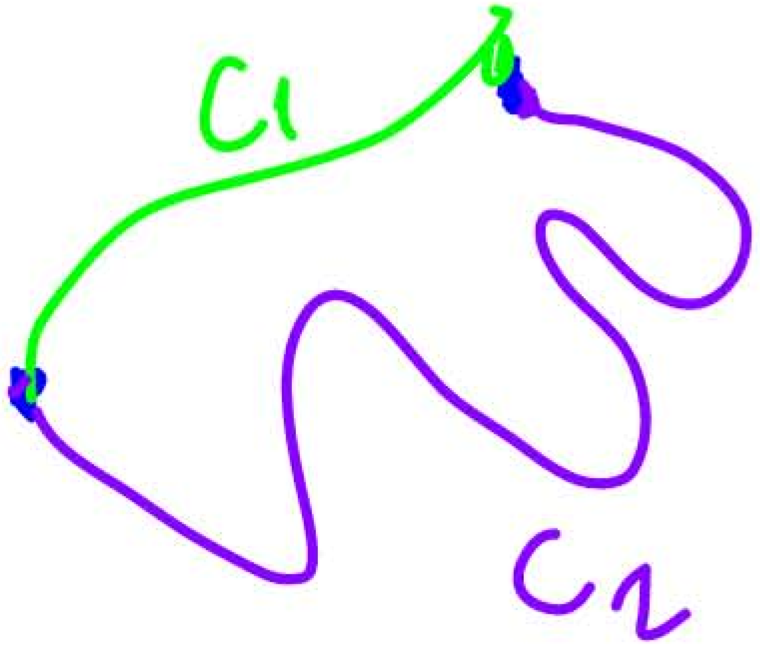
$$\quad P \quad Q$$

$$P_y = Q_x \; ?$$

$$1 = 1$$

$$\nabla f = F$$

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr = f(r(b)) - f(r(a))$$



integrate P

$$f = yx + h(y)$$

$$f_y = x + h'(y) = Q$$

$$= x$$

$$h'(y) = 0$$

$$h(y) = c$$

$$\boxed{f(x,y) = yx + c}$$

$f_x \; f_y$

$F = \langle P, Q \rangle$

Problem-Solving Stragegy: Finding a Potential Function for a Conservative Vector Field

$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$

1. Integrate P with respect to x. This results in a function of the form $g(x, y) + h(y)$, where $h(y)$ is unknown.
2. Take the partial derivative of $g(x, y) + h(y)$ with respect to y, which results in the function $g_y(x, y) + h'(y)$.
3. Use the equation $g_y(x, y) + h'(y) = Q(x, y)$ to find $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$.
5. Any function of the form $f(x, y) = g(x, y) + h(y) + C$, where C is a constant, is a potential function for $\mathbf{F}$.

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6.28 Find a potential function for $\mathbf{F}(x, y) = \langle \underbrace{e^x y^3 + y}_{P},\ \underbrace{3e^x y^2 + x}_{Q} \rangle . = \nabla f$

integrate with respect to x

$e^x y^3 + xy + h(y)$

diff. with respect to y

$3e^x y^2 + x + h'(y) = Q$

$h'(y) = 0$

$h(y) = c.$

$f(x,y) = e^x y^3 + xy + c$

Test: $P_y = Q_x$?

then F is conservative

there is a potential function f.

Theorem 6.10: The Cross-Partial Test for Conservative Fields

If $\mathbf{F} = \langle P, Q, R \rangle$ is a vector field on an open, simply connected region D and $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D, then $\mathbf{F}$ is conservative.

Theorem 6.11: Cross-Partial Property of Conservative Fields

Let $\mathbf{F} = \langle P, Q, R \rangle$ be a vector field on an open, simply connected region D. Then $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$ throughout D if and only if $\mathbf{F}$ is conservative.

conservative $\Rightarrow P_y = Q_x,\ P_z = R_x,\ Q_z = R_y$

for open and simply connected.