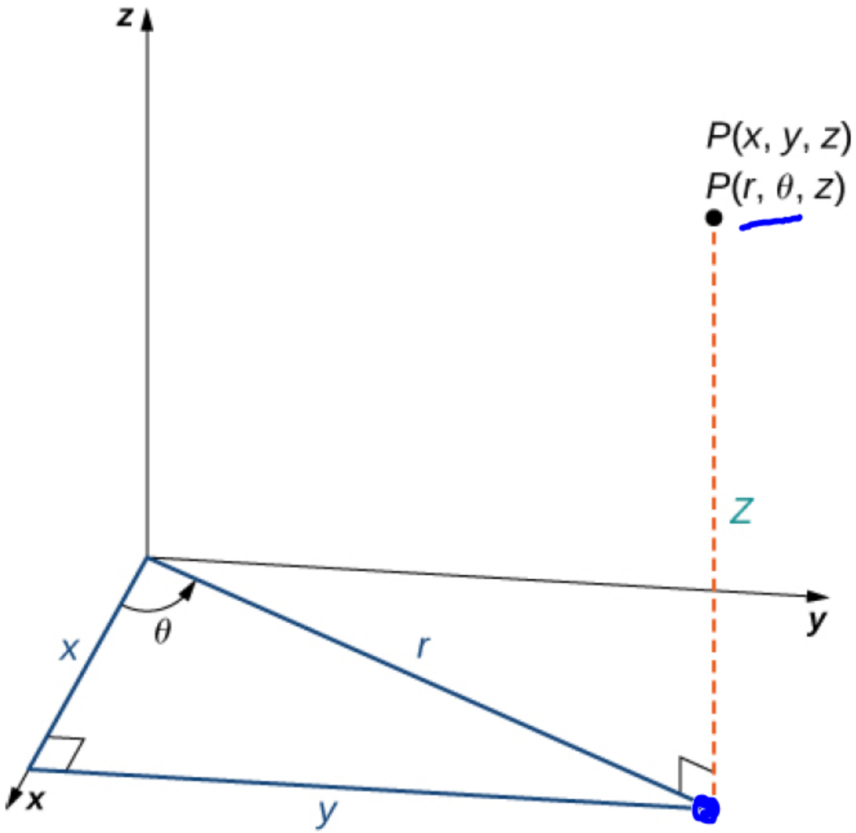
Cylindrical and spherical coordinates and review

Cylindrical Coordinates

Definition

In the **cylindrical coordinate system**, a point in space (**Figure 2.89**) is represented by the ordered triple (r, θ, z), where

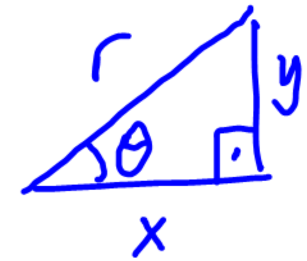
- (r, θ) are the polar coordinates of the point's projection in the xy-plane
- z is the usual z-coordinate in the Cartesian coordinate system

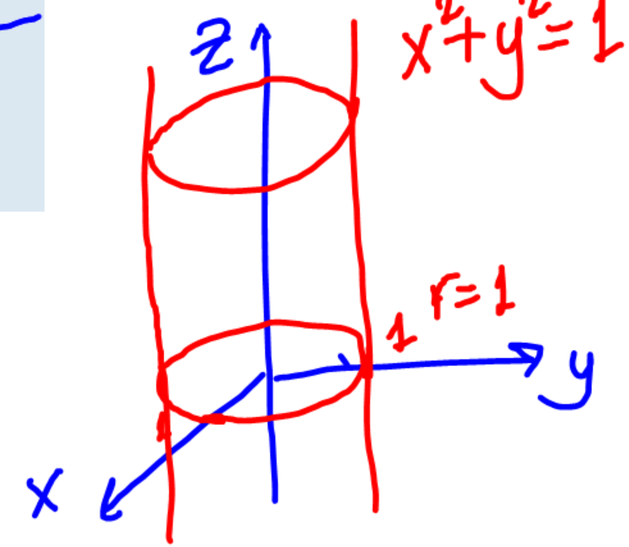


$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}$$

and

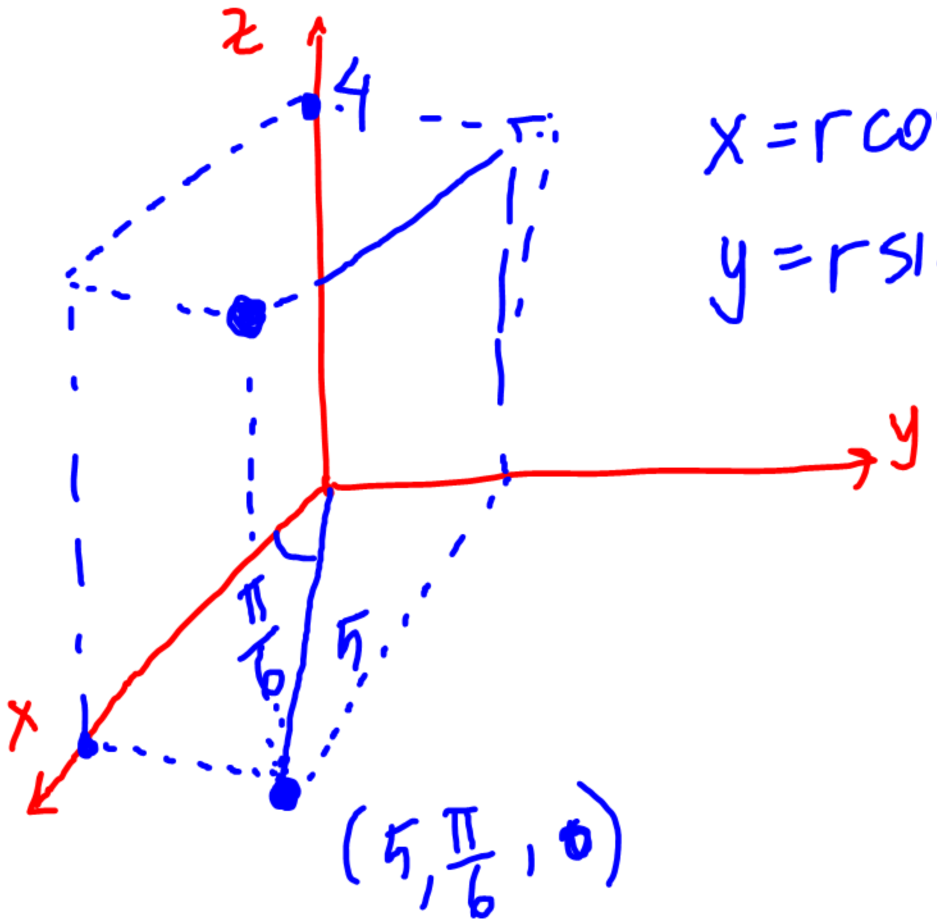
$$\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \\ z &= z \end{aligned}$$





2.55 Point R has cylindrical coordinates $\left(5, \frac{\pi}{6}, 4\right)$. Plot R and describe its location in space using rectangular, or Cartesian, coordinates.

$(r, \theta, 4)$



$$x = r\cos\theta = 5\cos\frac{\pi}{6} = 5\,\frac{\sqrt{3}}{2}$$

$$y = r\sin\theta = 5\sin\frac{\pi}{6} = 5/2$$

$$\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 4\right)$$

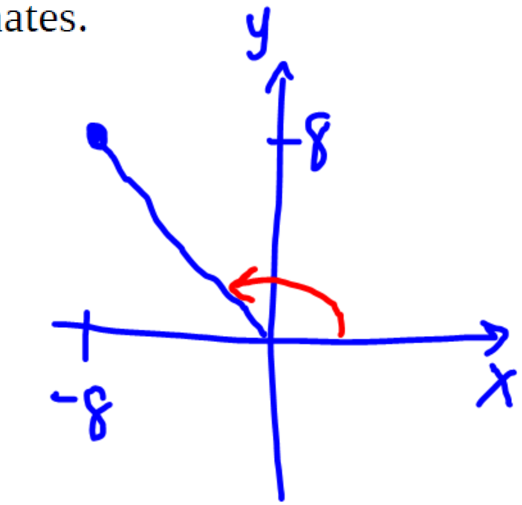
2.56 Convert point $(-8, 8, -7)$ from Cartesian coordinates to cylindrical coordinates.

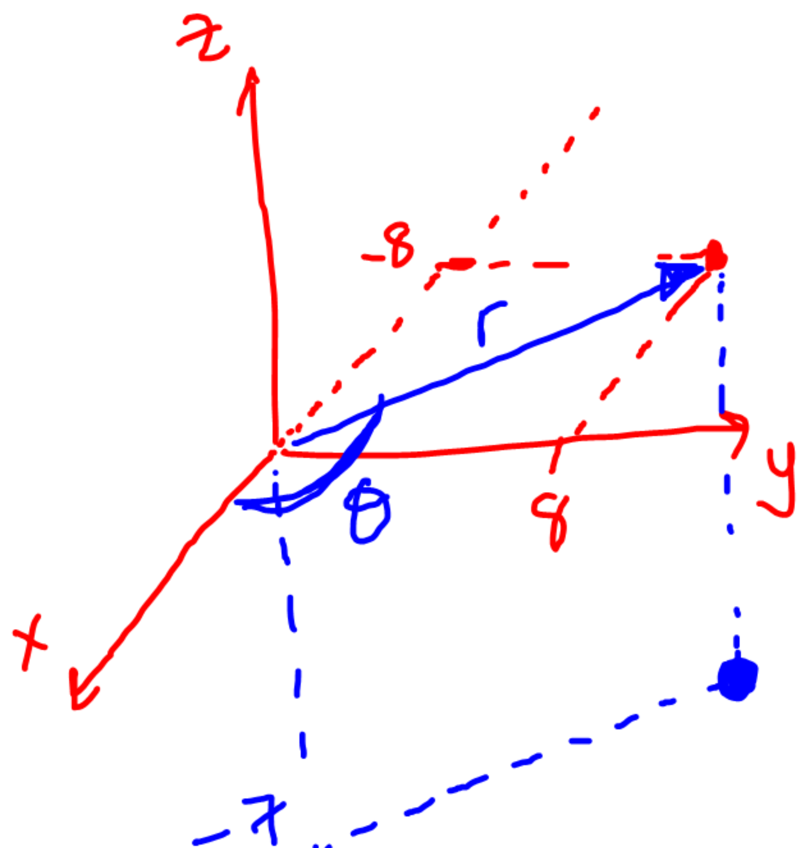
$x \quad y \quad z$

$r^2 = x^2 + y^2 \qquad r^2 = 64 + 64 = 128 \qquad r = 8\sqrt{2}$

$\tan\theta = \dfrac{y}{x} \qquad \tan\theta = \dfrac{8}{-8} = -1 \qquad \theta = \dfrac{3\pi}{4} = 135^\circ$

$\theta = \dfrac{7\pi}{4} = 315^\circ$

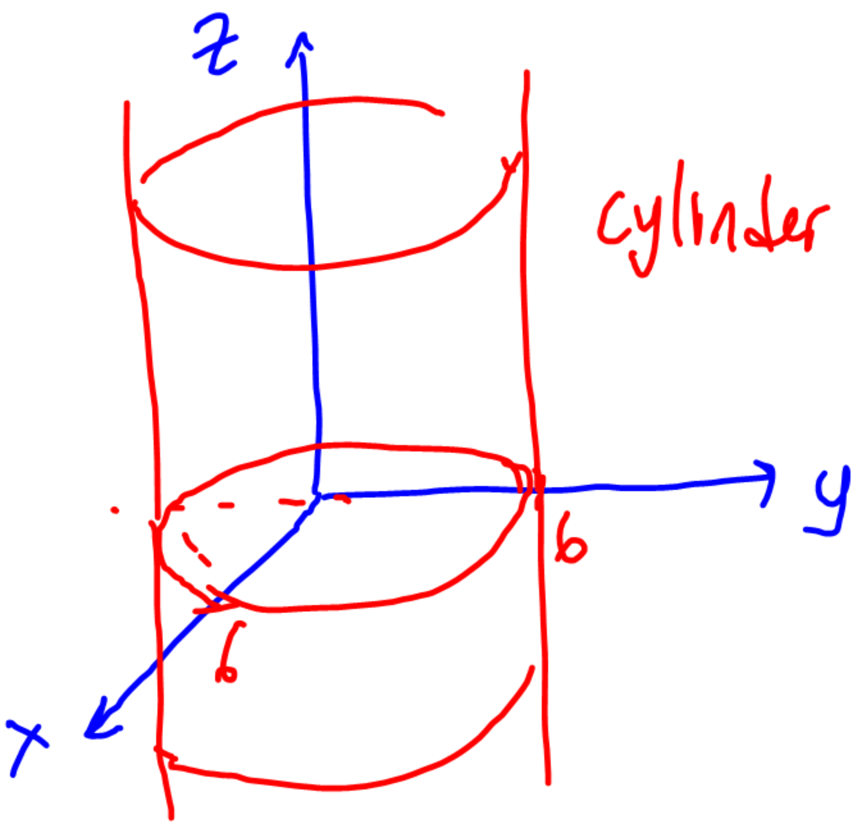




$\left(8\sqrt{2}, \dfrac{3\pi}{4}, -7\right)$

2.57 Describe the surface with cylindrical equation $r = 6$.

$x^2+y^2=6^2$



cylinder of radius 6

Spherical Coordinates

Definition

In the **spherical coordinate system**, a point P in space (Figure 2.97) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates; theta
- φ (the Greek letter phi) is the angle formed by the positive z-axis and line segment $\overline{OP}$, where O is the origin and $0 \leq \varphi \leq \pi$.

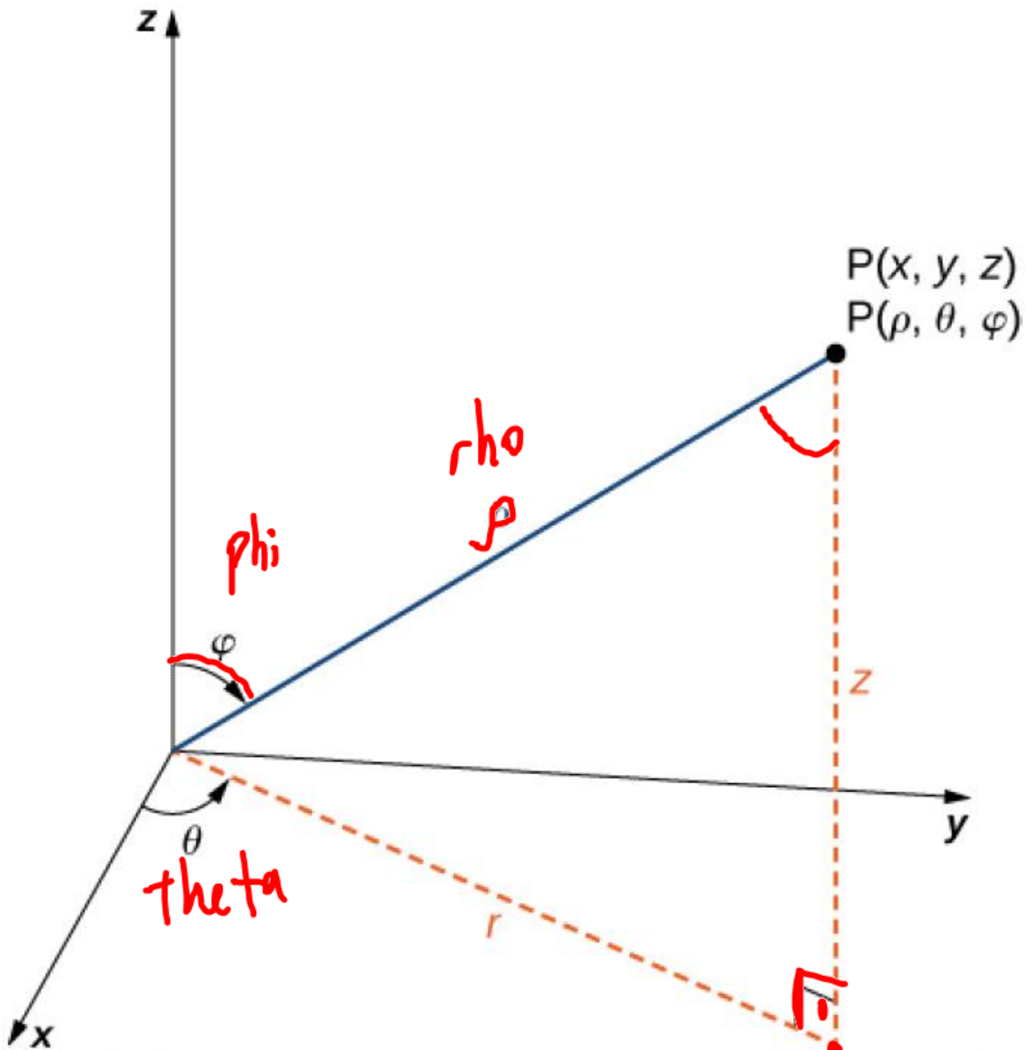
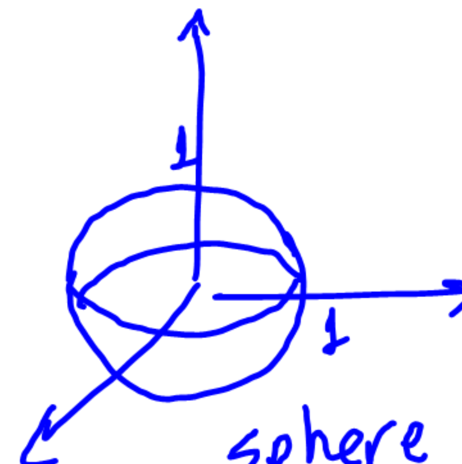


Figure 2.97 The relationship among spherical, rectangular, and cylindrical coordinates.

$$\tan\theta = \frac{y}{x}$$

$$r^2 = x^2 + y^2$$

$$\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$$



$\rho = 1$

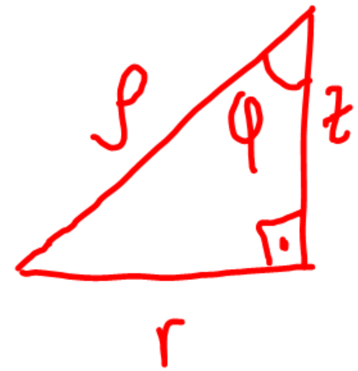
$x^2 + y^2 + z^2 = 1^2$

sphere of radius 1.

Theorem 2.16: Converting among Spherical, Cylindrical, and Rectangular Coordinates

Rectangular coordinates (x, y, z) and spherical coordinates (ρ, θ, φ) of a point are related as follows:

$$
\begin{aligned}
x &= \rho \sin\varphi \cos\theta \quad = r\cos\theta \\
y &= \rho \sin\varphi \sin\theta \\
z &= \rho \cos\varphi \\
&\text{and} \\
\rho^2 &= x^2 + y^2 + z^2 \\
\tan\theta &= \frac{y}{x} \\
\varphi &= \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right).
\end{aligned}
$$



$r = \rho \sin\varphi$

$z = \rho \cos\varphi$

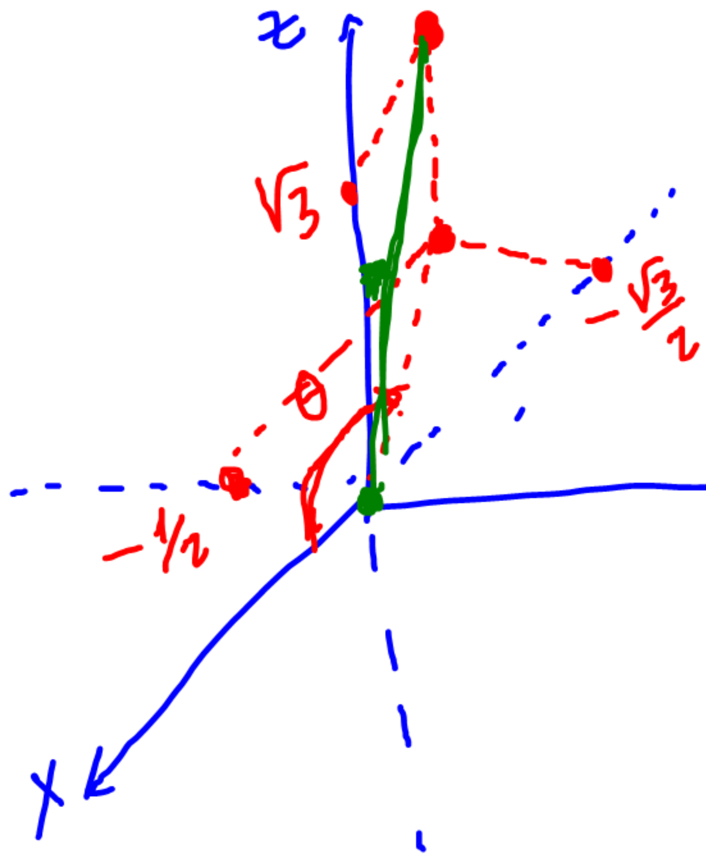
$\cos\varphi = \dfrac{z}{\rho}$

cylindrical ↔ spherical

$$
\begin{aligned}
r &= \rho \sin\varphi \\
\theta &= \theta \\
z &= \rho \cos\varphi \\
&\text{and} \\
\rho &= \sqrt{r^2 + z^2} \\
\theta &= \theta \\
\varphi &= \arccos\left(\frac{z}{\sqrt{r^2+z^2}}\right)
\end{aligned}
$$

2.58 Plot the point with spherical coordinates $\left(2,\ -\frac{5\pi}{6},\ \frac{\pi}{6}\right)$ and describe its location in both rectangular and cylindrical coordinates.

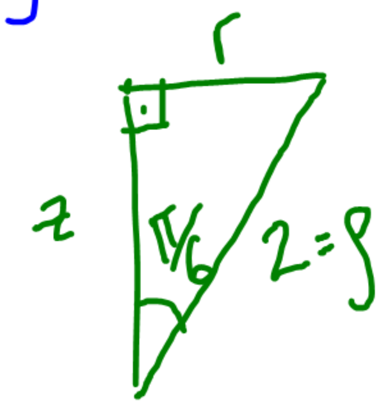
$\rho \quad \theta \quad \varphi$



$$x = 2\sin\frac{\pi}{6}\cos\left(-\frac{5\pi}{6}\right) = 2\,\frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}/2$$

$$y = 2\sin\frac{\pi}{6}\sin\left(-\frac{5\pi}{6}\right) = 2\,\frac{1}{2}\cdot\left(-\frac{1}{2}\right) = -1/2$$

$$z = 2\cos\frac{\pi}{6} = 2\frac{\sqrt{3}}{2} = \sqrt{3}$$



$$r = \rho\sin\varphi = 2\sin\frac{\pi}{6} = 2\cdot\frac{1}{2} = 1$$

$$\theta = \theta$$

$$z = \rho\cos\varphi = \sqrt{3}$$

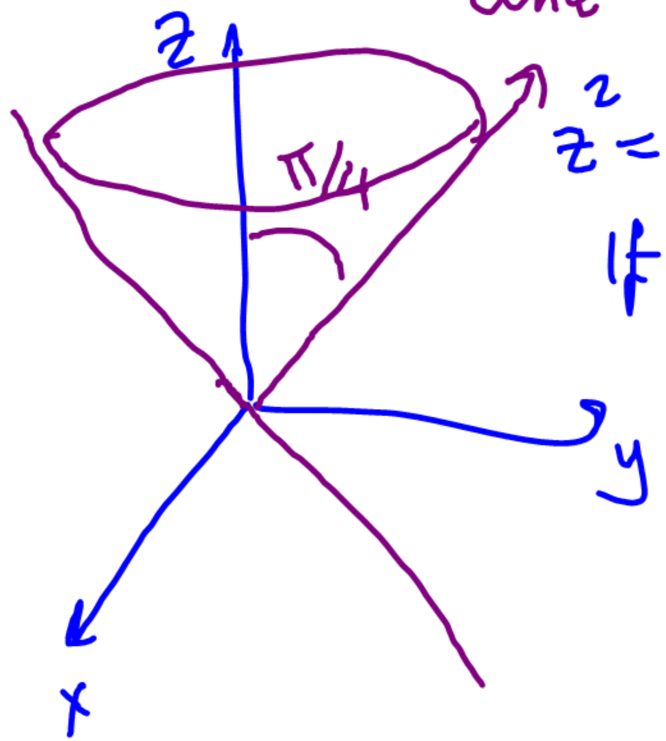
2.59 Describe the surfaces defined by the following equations.

a. $\rho = 13$ sphere of radius 13 $x^2+y^2+z^2=13^2$

b. $\theta = \frac{2\pi}{3}$ $\tan\theta = \frac{y}{x}$ $\tan\frac{2\pi}{3} = -\sqrt{3} = \frac{y}{x}$ $y = -\sqrt{3}x$ plane

c. $\varphi = \frac{\pi}{4}$

Cone



$z^2 = x^2 + y^2$

If $x=0$ $z = \mp y$

If $y=0$ $z = \mp x$

$z = \rho\cos\varphi$

$r = \rho\sin\varphi$

$\cos\varphi = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$\frac{1}{2} = \frac{z^2}{x^2+y^2+z^2}$

$\boxed{z^2 = x^2 + y^2}$

For the following exercises, the cylindrical coordinates (r, θ, z) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

364. $\left(3, \frac{\pi}{3}, 5\right)$

r, θ, z

$x = r\cos\theta = 3\cos\frac{\pi}{3} = 3 \cdot \frac{1}{2} = \frac{3}{2}$

$y = r\sin\theta = 3\sin\frac{\pi}{3} = 3 \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 5\right)$

For the following exercises, the rectangular coordinates (x, y, z) of a point are given. Find the cylindrical coordinates (r, θ, z) of the point.

370. $(-2\sqrt{2}, 2\sqrt{2}, 4)$

x y z

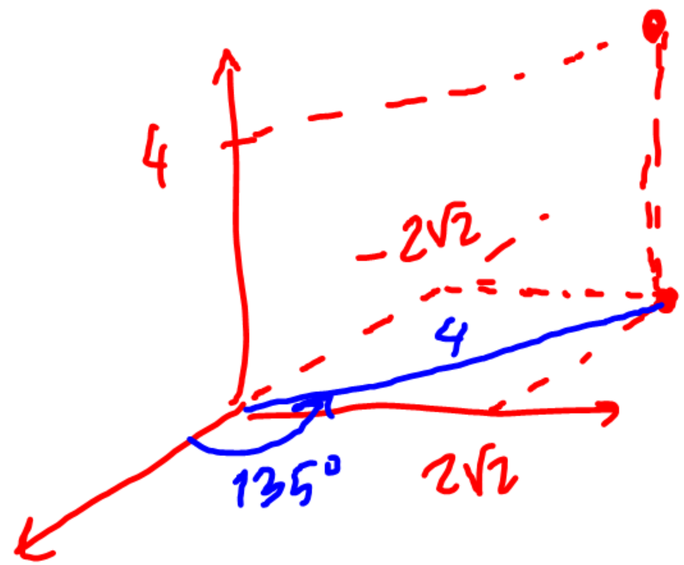
$$r = \sqrt{x^2+y^2} = \sqrt{(-2\sqrt{2})^2+(2\sqrt{2})^2} = \sqrt{16} = 4$$

$$\tan\theta = \frac{y}{x} = \frac{2\sqrt{2}}{-2\sqrt{2}} = -1$$

$\theta = \frac{3\pi}{4}$ ✓

$\theta = \frac{7\pi}{4}$

$$\left(4, \frac{3\pi}{4}, 4\right)$$



For the following exercises, the equation of a surface in rectangular coordinates is given. Find the equation of the surface in cylindrical coordinates.

382. $y = 2x^2$

$x = r\cos\theta$

$y = r\sin\theta$

$r\sin\theta = 2(r\cos\theta)^2$

$r\sin\theta = 2r^2\cos^2\theta$

$\frac{\sin\theta}{2\cos^2\theta} = r$

$r = \frac{1}{2} \cdot \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta}$

$r = \frac{\tan\theta \cdot \sec\theta}{2}$

For the following exercises, the spherical coordinates (ρ, θ, φ) of a point are given. Find the rectangular coordinates (x, y, z) of the point.

388. $\left(3, \frac{\pi}{4}, \frac{\pi}{6}\right)$

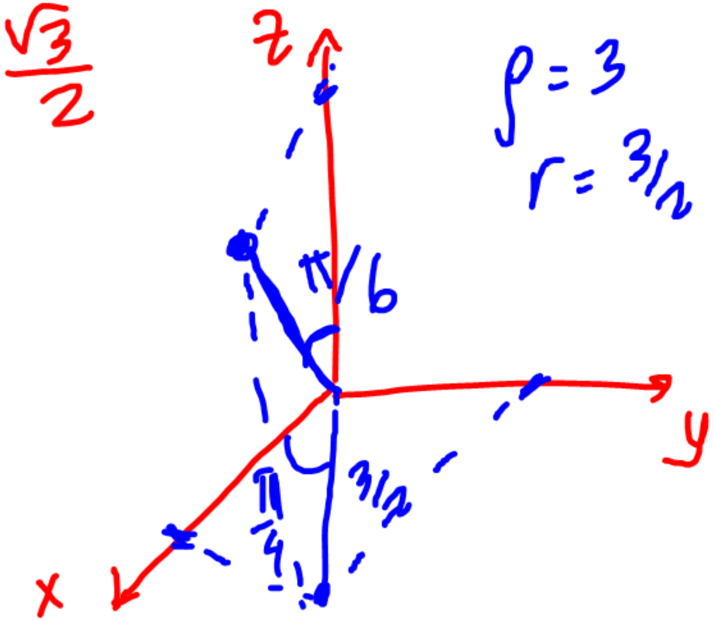
$\rho \quad \theta \quad \varphi$

$x = r\cos\theta = \rho\sin\varphi\cos\theta = 3\sin\frac{\pi}{6}\cdot\cos\frac{\pi}{4} = 3\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2}$

$y = r\sin\theta = \rho\sin\varphi\sin\theta = 3\sin\frac{\pi}{6}\cdot\sin\frac{\pi}{4} = 3\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2}$

$z = \rho\cos\varphi = 3\cdot\cos\frac{\pi}{6} = 3\frac{\sqrt{3}}{2}$

$\left(\frac{3\sqrt{2}}{4}, \frac{3\sqrt{2}}{4}, \frac{3\sqrt{3}}{2}\right)$



For the following exercises, find the vector and parametric equations of the line with the given properties.

435. The line that passes through points A $(1, 3, 5)$ and B $(-2, 6, -3)$

$\vec{u} = \overrightarrow{AB} = \vec{B} - \vec{A} = \langle -3, 3, -8 \rangle$ is a direction vector

$\langle x, y, z \rangle = \langle 1, 3, 5 \rangle + t\langle -3, 3, -8 \rangle$

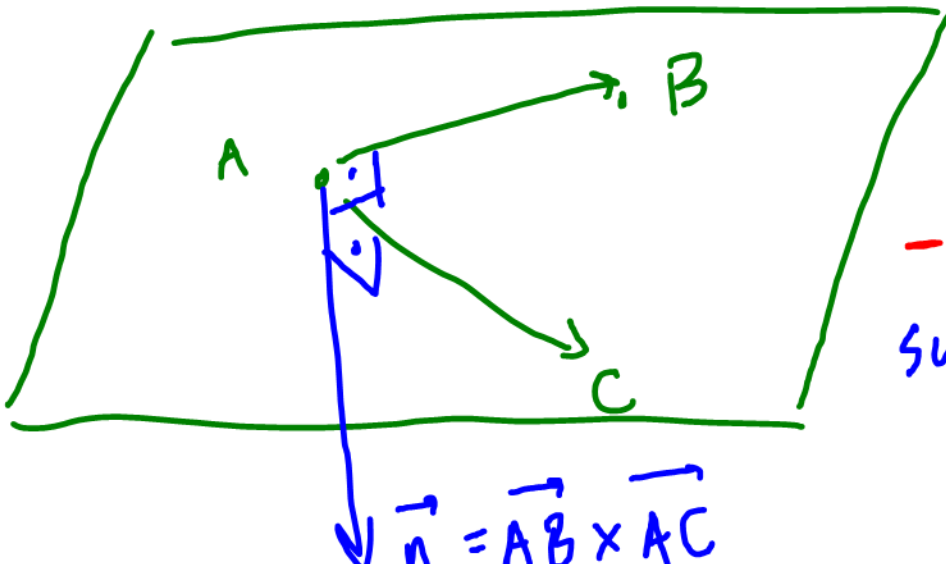
$$\left.\begin{aligned} x &= 1 - 3t \\ y &= 3 + 3t \\ z &= 5 - 8t \end{aligned}\right\} \text{parametric.}$$

$$t = \frac{x-1}{-3} = \frac{y-3}{3} = \frac{z-5}{-8} \quad \text{symmetric}$$

For the following exercises, find the equation of the plane with the given properties.

437. The plane that passes through points $(0, 1, 5)$, $(2, -1, 6)$, and $(3, 2, 5)$.

A $\qquad B \qquad C$



$\vec{n} = \vec{AB} \times \vec{AC}$

$\vec{AB} = \langle 2, -2, 1 \rangle$

$\vec{AC} = \langle 3, 1, 0 \rangle$

$$\vec{n} = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 3 & 1 & 0 \end{vmatrix} = -i - j(-3) + k(2+6)$$

$$= \langle -1, 3, 8 \rangle$$

$-1x + 3y + 8z + d = 0$

substitute $A(0, 1, 5)$

$3 + 40 + d = 0 \qquad d = -43$

$$\boxed{-x + 3y + 8z = 43}$$

For the following exercises, write the given equation in cylindrical coordinates and spherical coordinates.

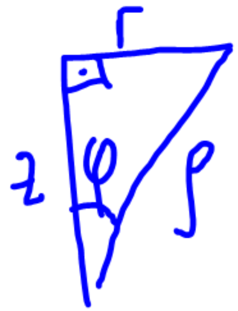
440. $x^2 + y^2 + z^2 = 144$

cylindrical $r^2 + z^2 = 12^2$

spherical $x^2+y^2+z^2 = \rho^2 = 12^2$ $\rho = 12$ sphere of radius 12.

For the following exercises, convert the given equations from cylindrical or spherical coordinates to rectangular coordinates. Identify the given surface.

442. $\rho^2\left(\sin^2(\varphi) - \cos^2(\varphi)\right) = 1$



$r = \rho\sin\varphi$

$z = \rho\cos\varphi$

$r^2 - z^2 = 1$

$x^2 + y^2 - z^2 = 1$

hyperboloid of one sheet

443. $r^2 - 2\overset{x}{r\cos(\theta)} + z^2 = 1$

$x^2 + y^2 - 2x + z^2 = 1$

$(x-1)^2 + y^2 + z^2 = 2$

ellipsoid

sphere of radius $\sqrt{2} = \rho$

centered at $(1,0,0)$

447. Calculate the work done by moving a particle from position $_A(1, 2, 0)$ to $_B(8, 4, 5)$ along a straight line with a force $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.

$$\underbrace{F}_{\text{force}} \cdot \underbrace{\overrightarrow{AB}}_{\text{displacement}} = \langle 2, 3, -1 \rangle \cdot \langle 7, 2, 5 \rangle$$

$$= 14 + 6 - 5 = \underline{\underline{15}} \text{ unit.}$$

$$\overrightarrow{AB} = \langle 7, 2, 5 \rangle$$