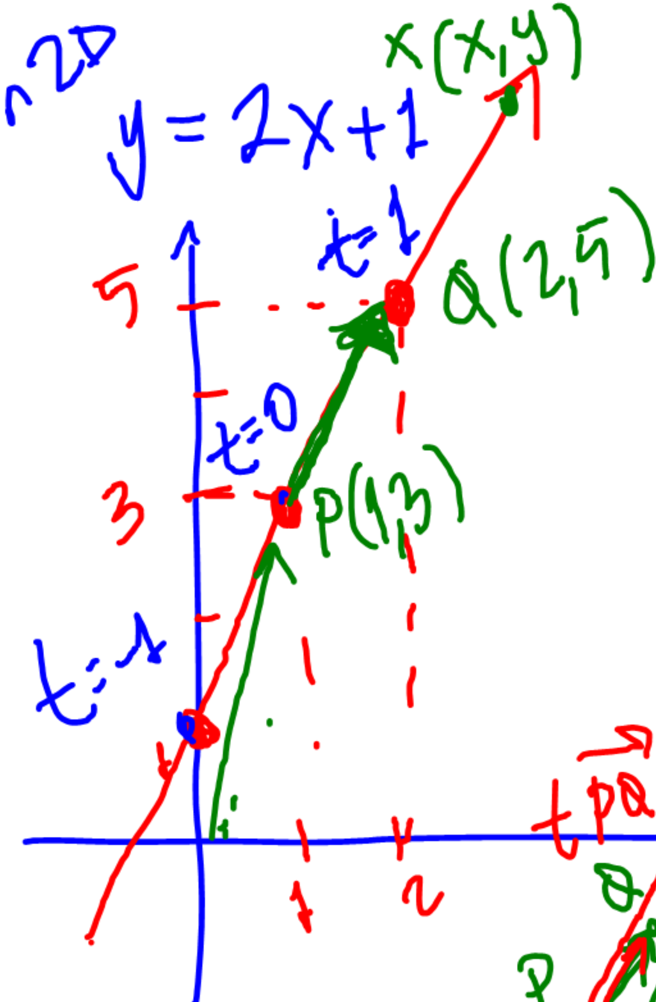
Lines and planes in space

in 2D

$y = 2x + 1$

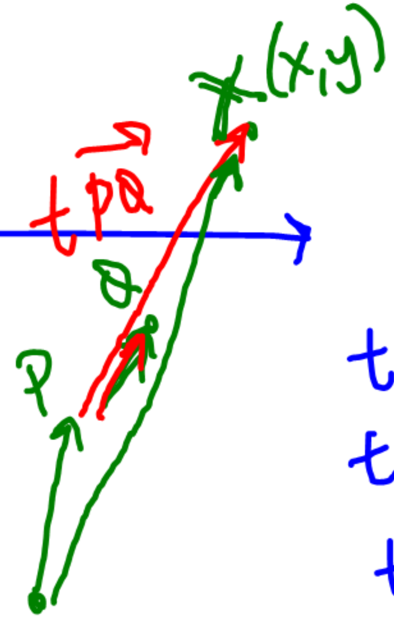


$\vec{PQ} = \langle 1, 2 \rangle$

any point on the line can be written as

$\langle x, y \rangle = \vec{P} + t\,\vec{PQ} = \vec{X}$ vector form

$= \langle 1, 3 \rangle + t\,\langle 1, 2 \rangle = \langle 1+t, 3+2t \rangle$



$x = 1 + t$
$y = 3 + 2t$ parametric form

$t = x - 1 = \dfrac{y-3}{2}$

symmetric form.

$t = -1, \quad (0,1)$

$t = 0 \quad (1,3)$

$t = 1 \quad (2,5)$

$$\begin{aligned}
\overrightarrow{PQ} = \langle\, x - x_0,\ y - y_0,\ z - z_0 \,\rangle &= \langle\, ta,\ tb,\ tc \,\rangle \\
\langle\, x,\ y,\ z \,\rangle - \langle\, x_0,\ y_0,\ z_0 \,\rangle &= t\,\langle\, a,\ b,\ c \,\rangle \\
\langle\, x,\ y,\ z \,\rangle &= \langle\, x_0,\ y_0,\ z_0 \,\rangle + t\,\langle\, a,\ b,\ c \,\rangle .
\end{aligned}$$

Setting $\mathbf{r} = \langle\, x,\ y,\ z \,\rangle$ and $\mathbf{r}_0 = \langle\, x_0,\ y_0,\ z_0 \,\rangle$, we now have the **vector equation of a line**:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

$y = n + mx$

slope is replaced by direction vector.

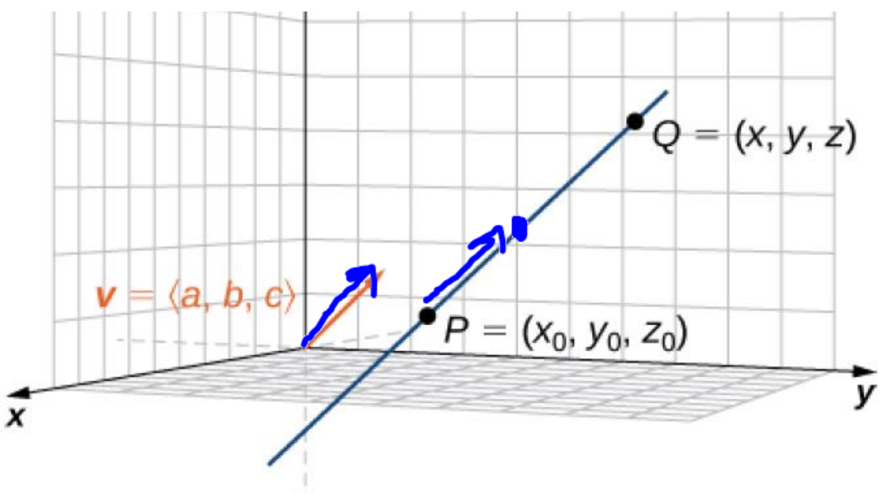
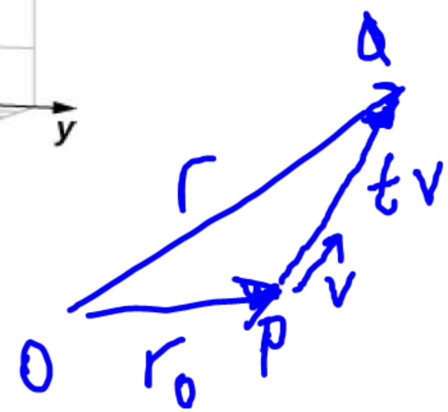


Figure 2.63 Vector $\mathbf{v}$ is the direction vector for $\overrightarrow{PQ}$.



$r = r_0 + tv$ vector form

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle$$
$$= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\}$$ parametric form

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

symmetric form

Theorem 2.11: Parametric and Symmetric Equations of a Line

A line L parallel to vector $\mathbf{v} = \langle a, b, c \rangle$ and passing through point $P(x_0, y_0, z_0)$ can be described by the following parametric equations:

$$x = x_0 + ta, \ y = y_0 + tb, \ \text{and } z = z_0 + tc. \tag{2.13}$$

If the constants $a, b,$ and c are all nonzero, then L can be described by the symmetric equation of the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{2.14}$$

2.43 Find parametric and symmetric equations of the line passing through points $\overset{A}{(1, -3, 2)}$ and $\overset{B}{(5, -2, 8)}$.

$$\vec{V} = \vec{AB} = \vec{B} - \vec{A} = \langle 5-1, -2-(-3), 8-2 \rangle = \langle 4, 1, 6 \rangle$$ a direction vector

$$\langle x, y, z \rangle = \langle 1, -3, 2 \rangle + t\langle 4, 1, 6 \rangle$$ vector form

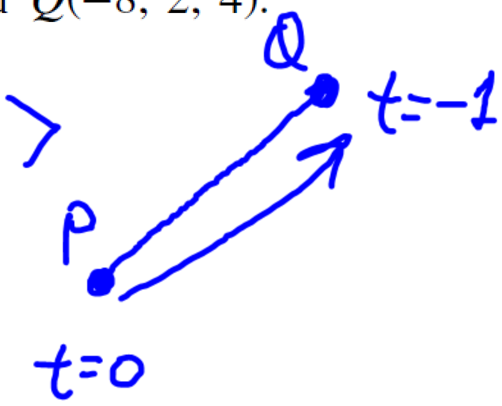
$$= \langle 1+4t, -3+t, 2+6t \rangle$$

$$\left.\begin{aligned} x &= 1+4t \\ y &= -3+t \\ z &= 2+6t \end{aligned}\right\}$$ parametric equations.

$$t = \frac{x-1}{4} = \frac{y+3}{1} = \frac{z-2}{6}$$

symmetric equations.

2.44 Find parametric equations of the line segment between points $P(-1, 3, 6)$ and $Q(-8, 2, 4)$.



$V=\overrightarrow{PQ} = \langle -7, -1, -2 \rangle$ OR $v = \overrightarrow{QP} = \langle 7, 1, 2 \rangle$

$\langle x, y, z \rangle = \underbrace{\langle -1, 3, 6 \rangle}_{P} + t \langle 7, 1, 2 \rangle$ line

$x = -1 + 7t,$

$y = 3 + t,$

$z = 6 + 2t,$

$-1 \le t \le 0$

$t = 0$ is P

$t = -1$ is Q

$\vec{P} + t\overrightarrow{PQ}, \quad 0 \le t \le 1$

$\langle -1, 3, 6 \rangle + t \langle -7, -1, -2 \rangle$

$t=0$ is P

$t=1$ is Q

$x = -1 - 7t$

$y = 3 - t \quad\quad 0 \le t \le 1$

$z = 6 - 2t$

Distance between a Point and a Line

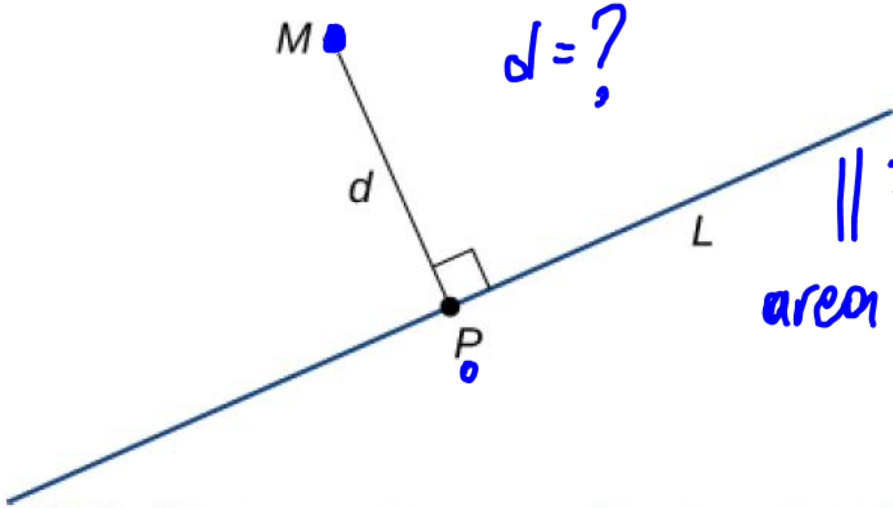


Figure 2.64 The distance from point M to line L is the length of $\overline{MP}$.

$\| \overrightarrow{PM} \times \vec{v} \| = \|v\| \cdot d$

area of parallelogram

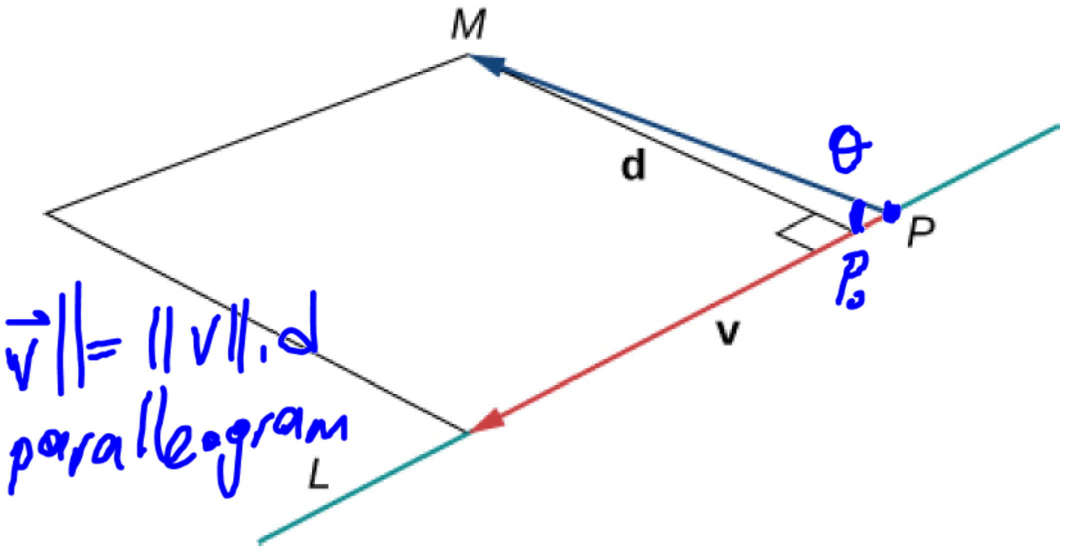


Figure 2.65 Vectors $\overrightarrow{PM}$ and $\mathbf{v}$ form two sides of a parallelogram with base $\| \mathbf{v} \|$ and height d, which is the distance between a line and a point in space.

Theorem 2.12: Distance from a Point to a Line

Let L be a line in space passing through point P with direction vector $\mathbf{v}$. If M is any point not on L, then the distance from M to L is

$$d = \frac{\| \overrightarrow{PM} \times \mathbf{v} \|}{\| \mathbf{v} \|}.$$

2.45 Find the distance between point $M(0, 3, 6)$ and the line with parametric equations $x = 1 - t,\ y = 1 + 2t,\ z = 5 + 3t$.

$$\langle x, y, z\rangle = \underbrace{\langle 1, 1, 5\rangle}_{P} + t\underbrace{\langle -1, 2, 3\rangle}_{v}$$

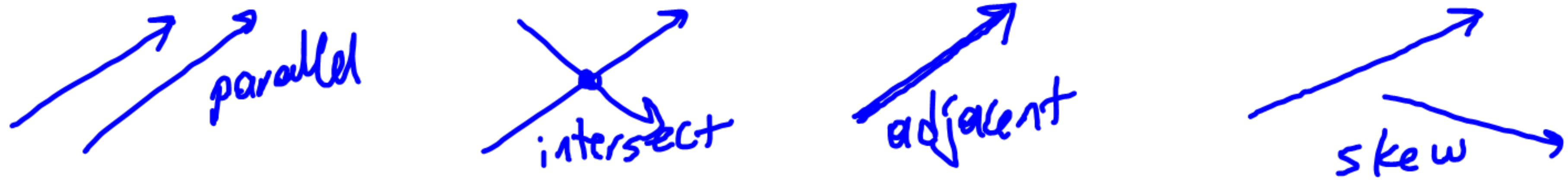
direction vector

$$v = \langle -1, 2, 3\rangle$$

$$\vec{PM} = \langle -1, 2, 1\rangle$$

$$\vec{PM} \times \vec{v} = \begin{vmatrix} i & j & k \\ -1 & 2 & 1 \\ -1 & 2 & 3 \end{vmatrix} = i\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} - j\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} + k\begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} = 4i + 2j + 0k$$

$$d = \frac{\|\vec{PM} \times v\|}{\|v\|} = \frac{\sqrt{4^2 + 2^2}}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{\sqrt{20}}{\sqrt{14}} = \sqrt{\frac{10}{7}} = \frac{\sqrt{70}}{7} \text{ unit}$$



2.46 Describe the relationship between the lines with the following parametric equations:

$$x = 1 - 4t,\ y = 3 + t,\ z = 8 - 6t \qquad V_1 = \langle -4, 1, -6 \rangle$$

$$x = 2 + 3s,\ y = 2s,\ z = -1 - 3s. \qquad V_2 = \langle 3, 2, -3 \rangle$$

try common solution

$1 - 4t = 2 + 3s$

$3 + t = 2s$

$8 - 6t = -1 - 3s$

$$\begin{array}{r} 1 - 4t = 2 + 3s \\ + \ 12 + 4t = 8s \\ \hline 13 = 2 + 11s \end{array}$$

$1 = s$

$3 + t = 2$

$t = -1$

$(5, 2, 14)$

$(5, 2, -4)$

$8 - 6(-1) = -1 - 3 \times 1$

$14 = -4$ they do not intersect

the lines are skew to each other.

Equations for a Plane

$\mathbf{n} = \langle a, b, c \rangle$

$$\begin{aligned} \mathbf{n} \cdot \overrightarrow{PQ} &= 0 \\ \langle a, b, c \rangle \bullet \langle x - x_0, y - y_0, z - z_0 \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0. \end{aligned}$$

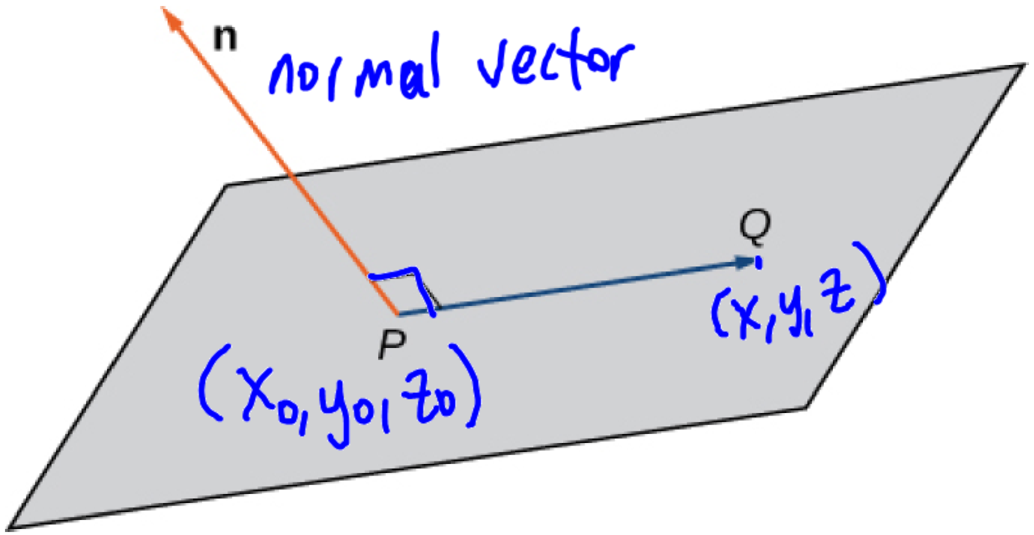


Figure 2.69 Given a point P and vector $\mathbf{n}$, the set of all points Q with $\overrightarrow{PQ}$ orthogonal to $\mathbf{n}$ forms a plane.

Definition

Given a point P and vector $\mathbf{n}$, the set of all points Q satisfying the equation $\mathbf{n} \cdot \overrightarrow{PQ} = 0$ forms a plane. The equation

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0 \tag{2.17}$$

is known as the vector equation of a plane.

The **scalar equation of a plane** containing point $P = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.} \tag{2.18}$$

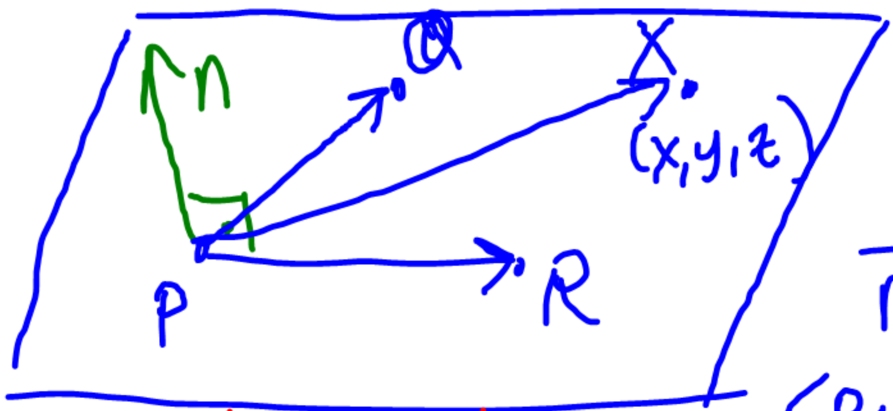
This equation can be expressed as $\boxed{ax + by + cz + d = 0,}$ where $d = -ax_0 - by_0 - cz_0$. This form of the equation is sometimes called the **general form of the equation of a plane**.

$ax+by+cz+d=0$

$n = \langle a, b, c \rangle$ normal vector

Find the general equation of the plane passing through P, Q, and R.

$P(1, 1, 1)$, $Q(2, 4, 3)$, and $R(-1, -2, -1)$



$$\vec{n} = \vec{PR} \times \vec{PQ} = \langle -2, -3, -2 \rangle \times \langle 1, 3, 2 \rangle$$

$$n = \begin{vmatrix} i & j & k \\ -2 & -3 & -2 \\ 1 & 3 & 2 \end{vmatrix} = i(-6+6) - j(-4+2) + k(-6+3)$$

$$= 2j - 3k = \langle \underset{a}{0}, \underset{b}{2}, \underset{c}{-3} \rangle$$

$$\vec{n} \cdot \vec{PX} = 0$$

$$\langle 0, 2, -3 \rangle \cdot \langle x-1, y-1, z-1 \rangle = 0$$

$$0(x-1) + 2(y-1) - 3(z-1) = 0$$

$$\underline{2y - 3z + 1 = 0}$$

OR $ax + by + cz + d = 0$

$$0x + 2y - 3z + d = 0$$

$P(1,1,1)$ will satisfy

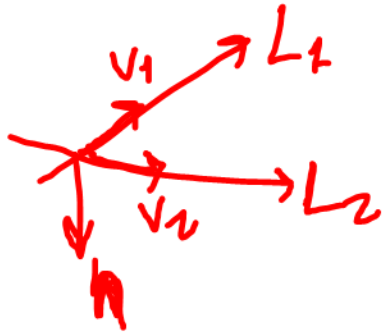
$$2 - 3 + d = 0$$

$$d = 1$$

2.47 Find an equation of the plane containing the lines L_1 and L_2:

direction vectors.

$n = v_1 \times v_2$



$L_1 : x = -y = z = t$

$L_2 : \dfrac{x-3}{2} = \dfrac{y}{1} = \dfrac{z-2}{1}.$

$v_1 = \langle 1, -1, 1 \rangle$

$v_2 = \langle 2, 1, 1 \rangle$

$$n = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = i(-1-1) - j(1-2) + k(1-(-2))$$

$$= -2i + j + 3k = \langle -2, 1, 3 \rangle$$

$-2x + y + 3z + d = 0$

$t = 0$, $(0,0,0)$ is on L_1

therefore on the plane

$d = 0$

$$\boxed{-2x + y + 3z = 0}$$

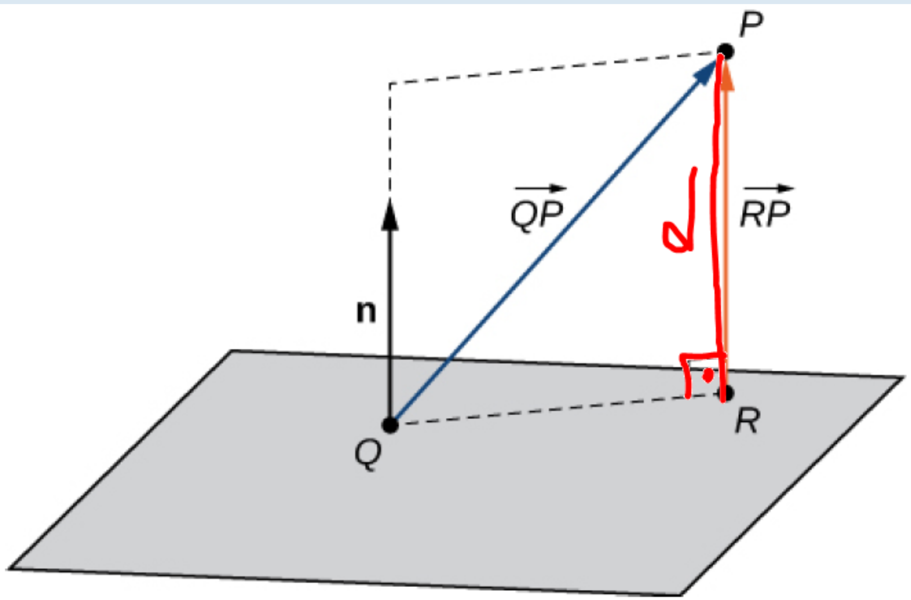
$(3,0,2)$ is on L_2

satisfies the equation.

Theorem 2.13: The Distance between a Plane and a Point

Suppose a plane with normal vector $\mathbf{n}$ passes through point Q. The distance d from the plane to a point P not in the plane is given by

$$d = \|\operatorname{proj}_{\mathbf{n}} \overrightarrow{QP}\| = \left|\operatorname{comp}_{\mathbf{n}} \overrightarrow{QP}\right| = \frac{\left|\overrightarrow{QP} \cdot \mathbf{n}\right|}{\|\mathbf{n}\|}. \tag{2.19}$$



Q is any point on the plane

Figure 2.70 We want to find the shortest distance from point P to the plane. Let point R be the point in the plane such that, for any other point in the plane Q, $\|\overrightarrow{RP}\| < \|\overrightarrow{QP}\|$.

$ax+by+cz+d=0$

2.48 Find the distance between point $P = (5, -1, 0)$ and the plane given by $4x + 2y - z = 3$.

$n = \langle 4, 2, -1 \rangle$ normal vector.

we need a point Q on the plane.

Let's say $x=0$, $y=2$, $4\times 0 + 2\times 2 - z = 3$, $z=1$

$Q(0,2,1)$ $\quad \overrightarrow{QP} = \langle 5, -3, -1 \rangle$

absolute value

$$d = \frac{\|\overrightarrow{QP} \cdot \vec{n}\|}{\|n\|} = \frac{\|\langle 5,-3,-1\rangle \cdot \langle 4,2,-1\rangle\|}{\sqrt{4^2+2^2+1^2}} = \frac{15}{\sqrt{21}} = \frac{5\sqrt{21}}{7}$$

Parallel and Intersecting Planes

2.49 Find parametric equations for the line formed by the intersection of planes $x + y - z = 3$ and $3x - y + 3z = 5$.

$n_1 = \langle 1, 1, -1 \rangle$

$n_2 = \langle 3, -1, 3 \rangle$

lets say $z=0$

$$\begin{aligned} x + y &= 3 \\ + \quad 3x - y &= 5 \\ \hline 4x &= 8 \end{aligned}$$

$x = 2$

$y = 1$

$(2,1,0)$ is a point on the line of intersection

$y=0$

$$\begin{aligned} 3/ \quad x - z &= 3 \\ + \quad 3x + 3z &= 5 \\ \hline 6x &= 14 \end{aligned}$$

$x = \frac{7}{3}$

$$v = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 3 & -1 & 3 \end{vmatrix} = 2i - 6j - 4k$$

$v = \langle 1, -3, -2 \rangle$

$\langle 2,1,0 \rangle + t\langle 1,-3,-2 \rangle$

$$\boxed{\begin{aligned} x &= 2 + t \\ y &= 1 - 3t \\ z &= -2t \end{aligned}}$$

line

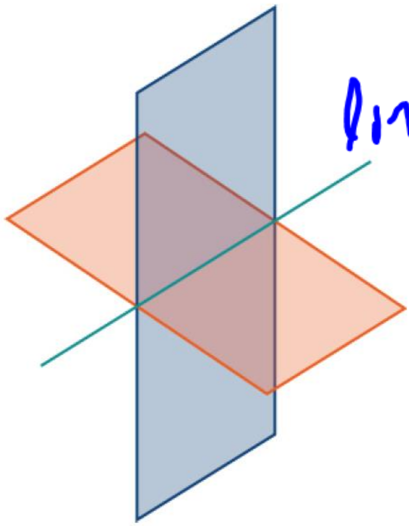


Figure 2.71 The intersection of two nonparallel planes is always a line.