4.2 | Limits and Continuity

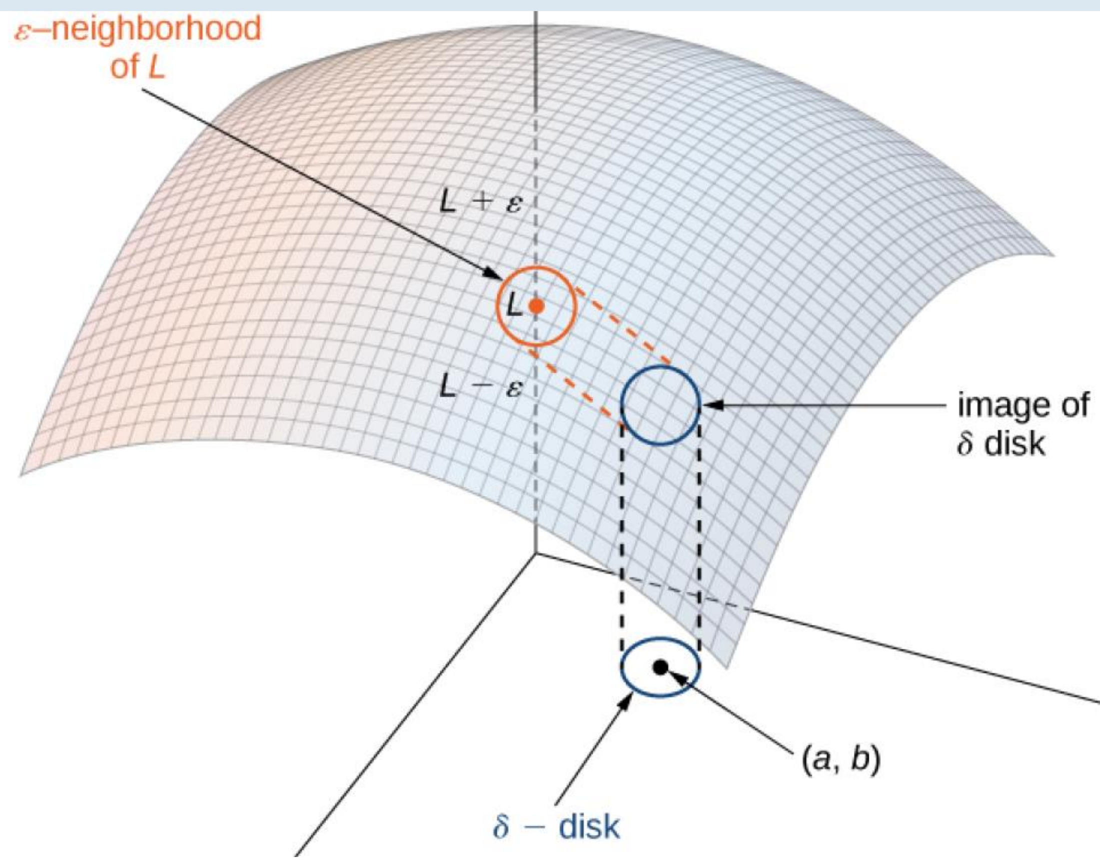
Definition

circle of radius δ

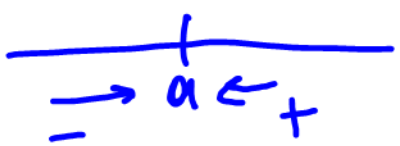
Consider a point $(a, b) \in \mathbb{R}^2$. A δ **disk** centered at point (a, b) is defined to be an open disk of radius δ centered at point (a, b)—that is,

$$\{(x, y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 < \delta^2\}$$

as shown in the following graph.



In Calculus 1



right limit

Definition formal definition of limit

Let f be a function of two variables, x and y. The limit of $f(x, y)$ as (x, y) approaches (a, b) is L, written

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for each $\varepsilon > 0$ there exists a small enough $\delta > 0$ such that for all points (x, y) in a δ disk around (a, b), except possibly for (a, b) itself, the value of $f(x, y)$ is no more than ε away from L (Figure 4.15). Using symbols, we write the following: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

Informally, this says $f(x,y)$ approaches to L (limit) as (x,y) approaches to (a,b)

Constant Law:

$$\lim_{(x,\ y) \to (a,\ b)} c = c$$

Identity Laws:

$$\lim_{(x,\ y) \to (a,\ b)} x = a$$

$$\lim_{(x,\ y) \to (a,\ b)} y = b$$

Sum Law:

$$\lim_{(x,\ y) \to (a,\ b)} (f(x,\ y) + g(x,\ y)) = L + M$$

Difference Law:

$$\lim_{(x,\ y) \to (a,\ b)} (f(x,\ y) - g(x,\ y)) = L - M$$

Constant Multiple l

$$\lim_{(x,\ y) \to (a,\ b)} (cf(x,\ y)) = cL$$

Product Law:

$$\lim_{(x,\ y) \to (a,\ b)} (f(x,\ y)g(x,\ y)) = LM$$

Quotient Law:

$$\lim_{(x,\ y) \to (a,\ b)} \frac{f(x,\ y)}{g(x,\ y)} = \frac{L}{M} \text{ for } M \neq 0$$

Power Law:

$$\lim_{(x,\ y) \to (a,\ b)} (f(x,\ y))^n = L^n$$

for any positive i

Root Law:

$$\lim_{(x,\ y) \to (a,\ b)} \sqrt[n]{f(x,\ y)} = \sqrt[n]{L}$$

61. $\displaystyle\lim_{(x,\,y)\to(1,\,2)} \frac{5x^2y}{x^2+y^2} = \frac{5.1^2.2}{1^2+2^2} = \frac{10}{5} = 2$

4.6 Evaluate the following limit:

$$\lim_{(x,\,y)\to(5,\,-2)} \sqrt[3]{\frac{x^2-y}{y^2+x-1}}. = \sqrt[3]{\frac{5^2-(-2)}{4+5-1}} = \sqrt[3]{\frac{27}{8}} = \frac{3}{2}$$

4.7 Show that

$$\lim_{(x,\,y)\to(2,\,1)} \frac{(x-2)(y-1)}{(x-2)^2+(y-1)^2} = \frac{0}{0}$$

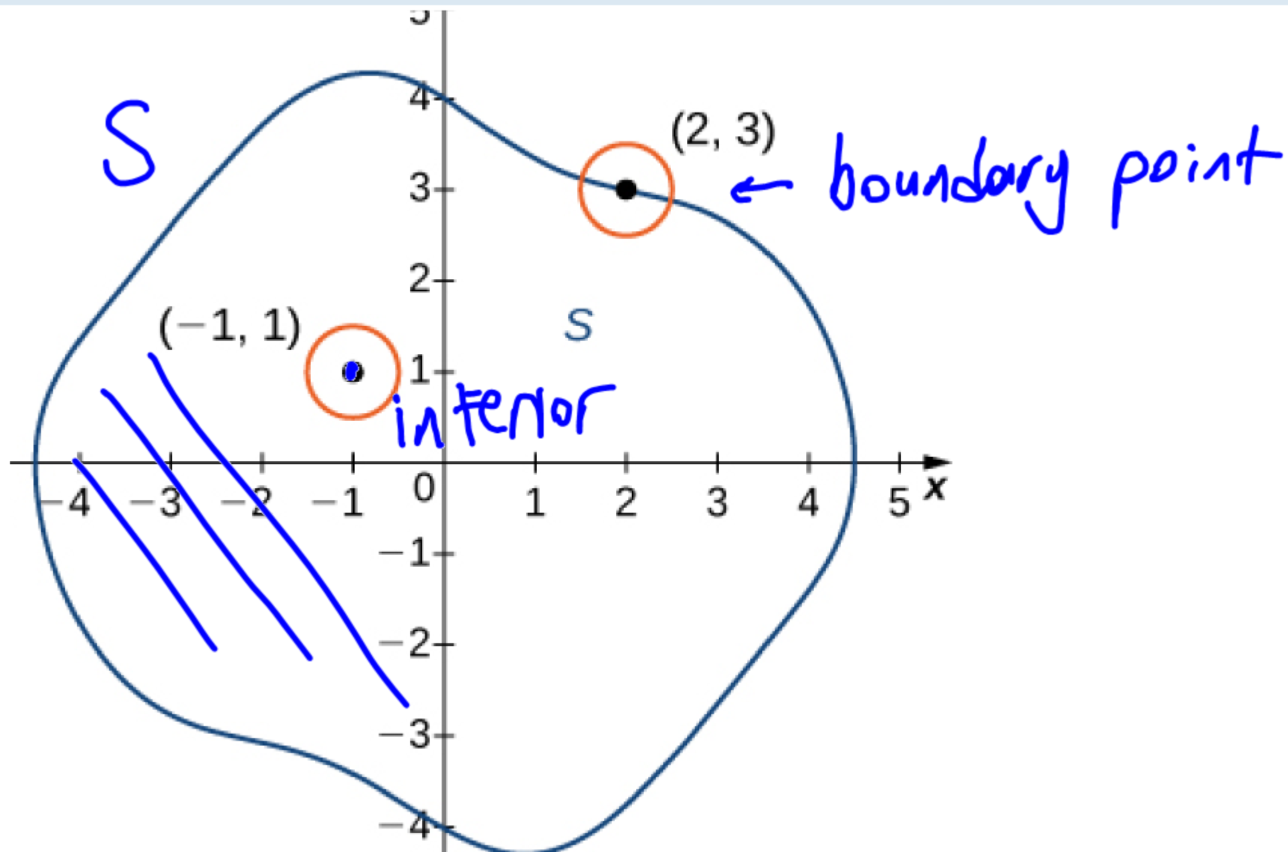
does not exist.

this limit does not exist.

Definition

Let S be a subset of $\mathbb{R}^2$ (**Figure 4.17**).

1. A point P_0 is called an **interior point** of S if there is a δ disk centered around P_0 contained completely in S.
2. A point P_0 is called a **boundary point** of S if every δ disk centered around P_0 contains points both inside and outside S.

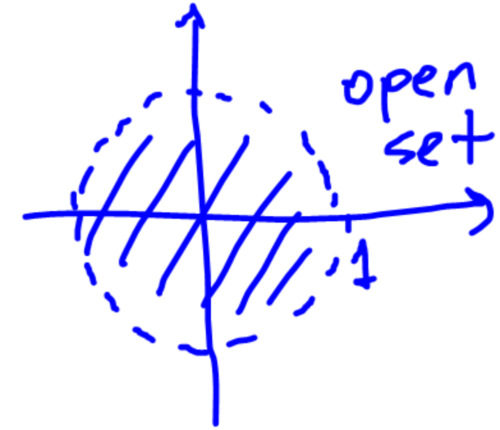


Definition

Let S be a subset of $\mathbb{R}^2$ (**Figure 4.17**).

1. S is called an **open set** if every point of S is an interior point.
2. S is called a **closed set** if it contains all its boundary points.

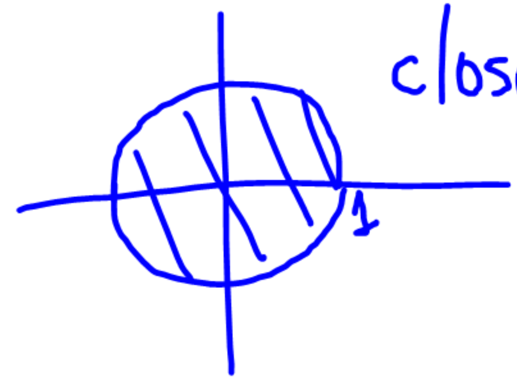
$x^2+y^2<1$



$(-1,1)$ open interval

$[-1,1]$ closed interval

$x^2+y^2 \leq 1$



closed set

Definition

Let S be a subset of $\mathbb{R}^2$ (Figure 4.17).

1. An open set S is a **connected set** if it cannot be represented as the union of two or more disjoint, nonempty open subsets.
2. A set S is a **region** if it is open, connected, and nonempty.

R

Definition

Let f be a function of two variables, x and y, and suppose (a, b) is on the boundary of the domain of f. Then, the limit of $f(x, y)$ as (x, y) approaches (a, b) is L, written

$$\lim_{(x, y) \to (a, b)} f(x, y) = L,$$

if for any $\varepsilon > 0$, there exists a number $\delta > 0$ such that for any point (x, y) inside the domain of f and within a suitably small distance positive δ of (a, b), the value of $f(x, y)$ is no more than ε away from L (**Figure 4.15**). Using symbols, we can write: For any $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon \text{ whenever } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$

4.8 Evaluate the following limit:

$$\lim_{(x,\ y) \to (5,\ -2)} \sqrt{29 - x^2 - y^2}.$$

$z = \sqrt{29 - x^2 - y^2}$

Domain

$\{(x,y) \mid x^2 + y^2 \leq 29\}$

$(5,-2)$ is a boundary point

$x^2 + y^2 + z^2 = 29$

$= \sqrt{29 - (5)^2 - (-2)^2}$

$= 0$

Definition

A function $f(x, y)$ is continuous at a point (a, b) in its domain if the following conditions are satisfied:

1. $f(a, b)$ exists.
2. $\lim_{(x, y) \to (a, b)} f(x, y)$ exists.
3. $\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$.

Continuity

$z^2+2x^2+y^2=26$ ellipsoid

4.9 Show that the function $f(x, y) = \sqrt{26 - 2x^2 - y^2}$ is continuous at point $(2, -3)$.

$f(2,-3) = \sqrt{26-2\cdot 2^2-(-3)^2} = \sqrt{9} = 3$

Domain $2x^2+y^2 \leq 26$

ellipse interior

$\lim_{(x,y)\to(2,-3)} f(x,y) = \sqrt{26-8-9} = 3 = f(2,-3)$

it is continuous at $(2,-3)$

The Sum of Continuous Functions Is Continuous

The Product of Continuous Functions Is Continuous

The Composition of Continuous Functions Is Continuous

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?

$f(x,y) = x^2 y - y^3 + \cos x \sin y$ is continuous

$g(x,y) = e^x + \ln x$ is continuous on $\underset{x}{(0,\infty)} \times \underset{y}{(-\infty,\infty)}$

$g(x,y) = yx^2 + \ln y$

$$h(x,y) = \frac{1}{y - x^3}$$ is not continuous when $y = x^3$

4.11 Find $\lim_{(x,\,y,\,z)\to(4,\,-1,\,3)} \sqrt{13 - x^2 - 2y^2 + z^2}$.

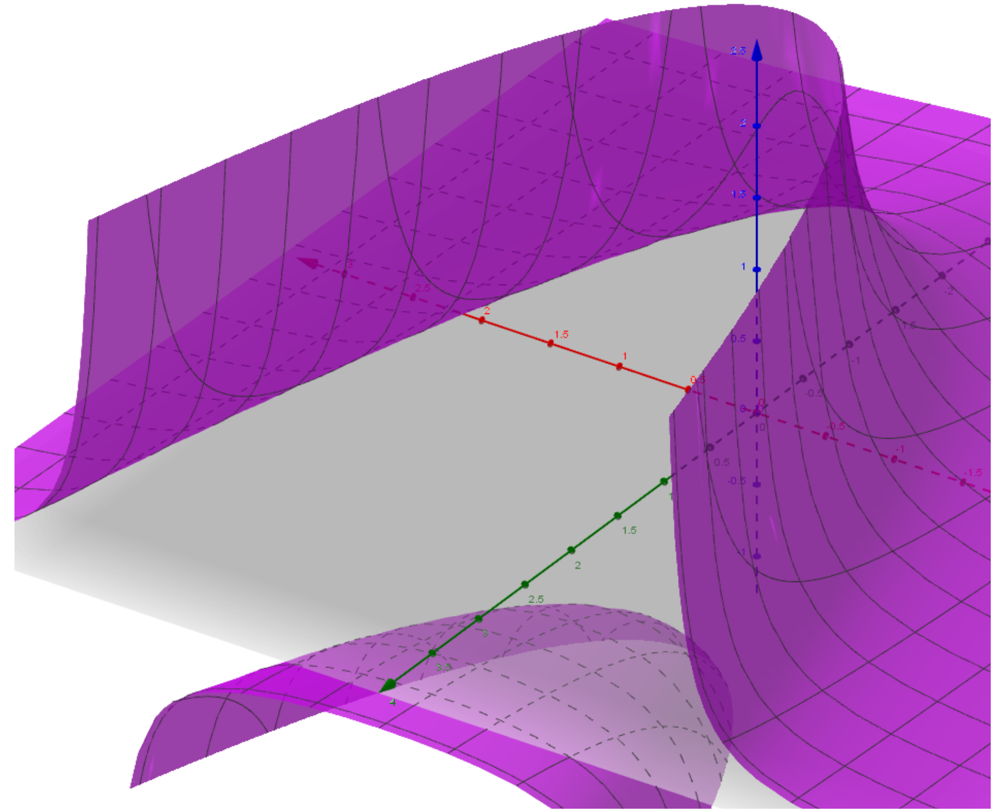
$$= \sqrt{13 - 4^2 - 2 + 3^2} = \sqrt{4} = 2$$

65. $\lim_{(x,\,y)\to(0,\,1)} \frac{y^2 \sin x}{x}$ $= \lim_{x\to 0} \frac{\sin x}{x} = 1$

98. Create a plot using graphing software to determine where the limit does not exist. Determine the region of the coordinate plane in which $f(x, y) = \dfrac{1}{x^2 - y}$ is continuous.

f is undefined on $\{(x,y) \mid y = x^2\}$

it is continuous except the points on the parabola $y = x^2$



4.3 | Partial Derivatives

Definition

Let $f(x, y)$ be a function of two variables. Then the **partial derivative** of f with respect to x, written as $\partial f/\partial x$, or f_x, is defined as

$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}. \tag{4.12}$$

The partial derivative of f with respect to y, written as $\partial f/\partial y$, or f_y, is defined as

$$f_y = \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}. \tag{4.13}$$

Calculus 1

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x,y) = x^2 - y^2 + yx - x$$

$$f_x = 2x + y - 1$$

$$f_y = -2y + x$$

4.13 Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the function $f(x, y) = \tan\left(x^3 - 3x^2y^2 + 2y^4\right)$ by holding the opposite variable constant, then differentiating.

for f_x consider y to be constant differentiate with respect to x.

$(\tan u)' = \sec^2 u \cdot u'$

$$f_x = \frac{\partial f}{\partial x} = \sec^2\left(x^3 - 3x^2y^2 + 2y^4\right)\left[3x^2 - 6xy^2\right] \leftarrow y \text{ is constant}$$

$$f_y = \frac{\partial f}{\partial y} = \sec^2\left(x^3 - 3x^2y^2 + 2y^4\right)\left[-6x^2y + 8y^3\right] \leftarrow x \text{ is constant}$$

$$\frac{\partial\left(x^3 - 3x^2y^2 + 2y^4\right)}{\partial y} = -6x^2y + 8y^3)$$

4.14 Use a contour map to estimate $\partial f/\partial y$ at point $(0, \sqrt{2})$ for the function

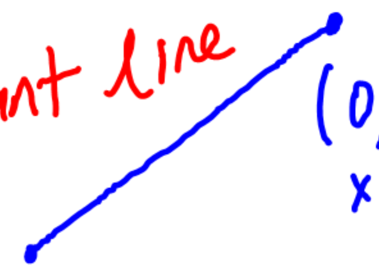
$$f(x, y) = x^2 - y^2.$$

Compare this with the exact answer.

$\sqrt{2}$ is close to $3/2$

$$\frac{f(0,\sqrt{2}) - f(0,\frac{3}{2})}{\sqrt{2}-\frac{3}{2}} \approx f_y(0,\sqrt{2})$$

secant line



$(0, \frac{3}{2}, f(0,\frac{3}{2}))$

x y z

$(0,\sqrt{2}, f(0,\sqrt{2}))$

$$= \frac{-2-(-\frac{9}{4})}{1.4-1.5} \approx \frac{\frac{1}{4}}{-0.1} = -2.5$$

$f(0,\sqrt{2}) = -2$

$f(0,\frac{3}{2}) = -\frac{9}{4}$

$f_y = -2y$

$f_y(0,\sqrt{2}) = -2\sqrt{2} \approx -2.8$ exact

Definition

Let $f(x, y, z)$ be a function of three variables. Then, the *partial derivative of* f *with respect to* x, written as $\partial f/\partial x$, or f_x, is defined to be

$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}. \tag{4.14}$$

The *partial derivative of* f *with respect to* y, written as $\partial f/\partial y$, or f_y, is defined to be

$$f_y = \frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k, z) - f(x, y, z)}{k}. \tag{4.15}$$

The *partial derivative of* f *with respect to* z, written as $\partial f/\partial z$, or f_z, is defined to be

$$f_z = \frac{\partial f}{\partial z} = \lim_{m \to 0} \frac{f(x, y, z+m) - f(x, y, z)}{m}. \tag{4.16}$$

$f(x,y,z) = x^2 z - yz^2 + z^3$

$f_x = 2xz$

$f_y = -z^2$

$f_z = x^2 - 2yz + 3z^2$

4.16 Calculate $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ for the function $f(x, y, z) = \sec\left(x^2 y\right)$

$f_z = 0$

$(\sec u)' = \sec u \tan u \cdot u'$

$f_x = \sec(x^2y)\tan(x^2y)\,2xy \quad , \quad \dfrac{\partial(x^2y)}{\partial x} = 2xy$

$f_y = \sec(x^2y)\tan(x^2y)\,x^2 \quad , \quad \dfrac{\partial(x^2y)}{\partial y} = x^2$

Higher-Order Partial Derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right], \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right], \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right], \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right].$$

f_{xx} $\quad$ f_{yx} $\quad$ f_{xy} $\quad$ f_{yy}

$f_{xy} = f_{yx}$

the order is not important

f_{xy}

$f_{xxy} = f_{yxx} = f_{xyx}$

$f(x,y) = x^3y^2 - x^3 + y^2$

$f_x = 3x^2y^2 - 3x^2$

$f_{xx} = 6xy^2 - 6x$

$f_{xy} = \frac{\partial(f_x)}{\partial y} = 6x^2y$

$f_y = 2x^3y + 2y$

$f_{yy} = 2x^3 + 2$

$f_{yx} = 6x^2y$

4.17 Calculate all four second partial derivatives for the function

$$f(x, y) = \sin(3x - 2y) + \cos(x + 4y).$$

$$f_x = \cos(3x-2y)\cdot 3 - \sin(x+4y)\cdot 1$$

$$f_y = \cos(3x-2y)\cdot(-2) - \sin(x+4y)\cdot 4$$

$$f_{xx} = -\sin(3x-2y)\cdot 9 - \cos(x+4y)$$

$$f_{xy} = -\sin(3x-2y)(-2)(3) - \cos(x+4y)\cdot 4$$

$$f_{yx} = -\sin(3x-2y)\cdot 3(-2) - \cos(x+4y)\cdot 4$$

$$f_{yy} = -\sin(3x-2y)(-2)(-2) - \cos(x+4y)\cdot 4\cdot 4$$

128. Let $f(x, y) = \frac{xy}{x-y}$. Find $f_x(2, -2)$ and $f_y(2, -2)$.

$$f_x = \frac{y(x-y) - (xy)1}{(x-y)^2}$$

quotient rule

$$f_x(2,-2) = \frac{-2(2+2)-(-4)}{4^2} = -\frac{1}{4}$$

$$f_y = \frac{x(x-y) - (xy)(-1)}{(x-y)^2}$$

$$f_y(2,-2) = \frac{2(2-(-2)) - (2(-2))(-1)}{16}$$

$$= \frac{8-4}{16} = \frac{1}{4}.$$

4.18 Verify that $u(x, y, t) = 2\sin\left(\frac{x}{3}\right)\sin\left(\frac{y}{4}\right)e^{-25t/16}$ is a solution to the heat equation

$$u_t = 9\left(u_{xx} + u_{yy}\right).$$