Vectors
dot product,
cross product

109. The points A, B, and C are collinear (in this order) if the relation $\| \overrightarrow{AB} \| + \| \overrightarrow{BC} \| = \| \overrightarrow{AC} \|$ is satisfied. Show that $A(5, 3, -1)$, $B(-5, -3, 1)$, and $C(-15, -9, 3)$ are collinear points.

$\overrightarrow{AB} = \vec{B} - \vec{A} = \langle -10, -6, 2 \rangle$

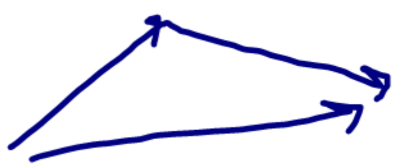
$\overrightarrow{BC} = \vec{C} - \vec{B} = \langle -10, -6, 2 \rangle$

$\overrightarrow{AC} = \vec{C} - \vec{A} = \langle -20, -12, 4 \rangle = 2\overrightarrow{AB}$

$\|\overrightarrow{AB}\| = \sqrt{140} = 2\sqrt{35} = \|\overrightarrow{BC}\|$

$\|\overrightarrow{AC}\| = \sqrt{400+144+16} = \sqrt{560}$

$= 2\sqrt{140} = 4\sqrt{35}$



110. Show that points $A(1, 0, 1)$, $B(0, 1, 1)$, and $C(1, 1, 1)$ are not collinear.

$\overrightarrow{AB} = \langle -1, 1, 0 \rangle$

$\overrightarrow{BC} = \langle 1, 0, 0 \rangle = i$

$\overrightarrow{AC} = \langle 0, 1, 0 \rangle = j$

$\|\overrightarrow{AB}\| + \|\overrightarrow{BC}\| = \sqrt{2} + 1 \neq \|\overrightarrow{AC}\| = 1$

so they are not on the same line.

Definition

The **dot product** of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by the sum of the products of the components

similarly on plane

$$\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3. \quad (2.3)$$

$u \cdot v = u_1 v_1 + u_2 v_2$

2.21 Find $\mathbf{u} \cdot \mathbf{v}$, where $\mathbf{u} = \langle 2, 9, -1 \rangle$ and $\mathbf{v} = \langle -3, 1, -4 \rangle$.

$u \cdot v = 2(-3) + 9 + (-1)(-4) = 7$

Theorem 2.3: Properties of the Dot Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors, and let c be a scalar.

i.	$\mathbf{u} \bullet \mathbf{v}$	$=$	$\mathbf{v} \bullet \mathbf{u}$	Commutative property
ii.	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$	$=$	$\mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$	Distributive property
iii.	$c(\mathbf{u} \cdot \mathbf{v})$	$=$	$(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$	Associative property
iv.	$\mathbf{v} \cdot \mathbf{v}$	$=$	$\lVert \mathbf{v} \rVert^2$	Property of magnitude

2.22 Find the following products for $\mathbf{p} = \langle 7, 0, 2 \rangle$, $\mathbf{q} = \langle -2, 2, -2 \rangle$, and $\mathbf{r} = \langle 0, 2, -3 \rangle$.

a. $(\mathbf{r} \cdot \mathbf{p})\mathbf{q}$ $= (0.7 + 2.0 + (-3)2)\, q = -6 \langle -2, 2, -2 \rangle = \langle 12, -12, 12 \rangle$

b. $\| \mathbf{p} \|^2$

$= p \cdot p = 7^2 + 2^2 = 53$

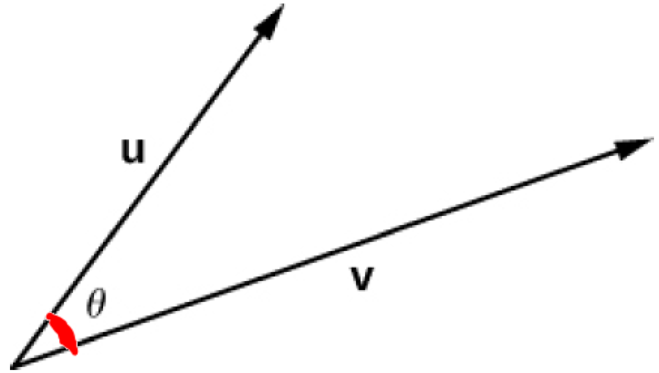


Figure 2.44 Let θ be the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ such that $0 \le \theta \le \pi$.

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \; \|\mathbf{v}\| \cos\theta.$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \; \|\mathbf{v}\|}$$

2.23 Find the measure of the angle, in radians, formed by vectors $\mathbf{a} = \langle 1, 2, 0 \rangle$ and $\mathbf{b} = \langle 2, 4, 1 \rangle$. Round to the nearest hundredth.

$$\frac{a \cdot b}{\|a\|\|b\|} = \frac{1 \cdot 2 + 2 \cdot 4 + 0 \cdot 1}{\sqrt{1^2+2^2}\sqrt{2^2+4^2+1^2}} = \frac{10}{\sqrt{5}\sqrt{21}} = \frac{2\sqrt{105}}{21} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{2}{21}\sqrt{105}\right) = 12.6^\circ \approx$$

radian

Theorem 2.5: Orthogonal Vectors

The nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal vectors** if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

$\cos\theta = 0$, $\theta = \frac{\pi}{2}$

2.24 For which value of x is $\mathbf{p} = \langle 2, 8, -1 \rangle$ orthogonal to $\mathbf{q} = \langle x, -1, 2 \rangle$?

we want

$p \cdot q = 0$ equivalent to $p \perp q$

$2x - 8 - 2 = 0$

when $\underline{x = 5}$

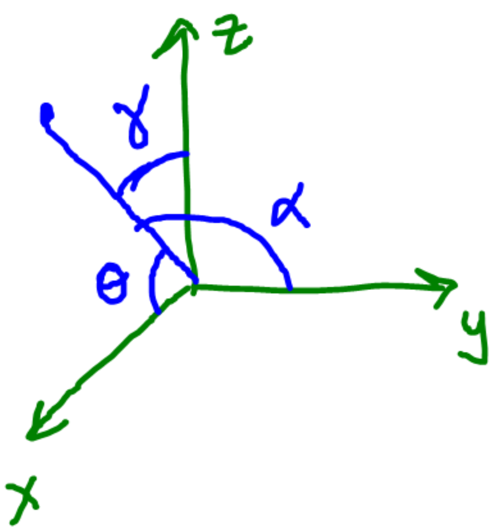
2.25 Let $\mathbf{v} = \langle 3, -5, 1 \rangle$. Find the measure of the angles formed by each pair of vectors.

a. $\mathbf{v}$ and $\mathbf{i}$ $\langle 1, 0, 0 \rangle$

b. $\mathbf{v}$ and $\mathbf{j}$ $\langle 0, 1, 0 \rangle$

c. $\mathbf{v}$ and $\mathbf{k}$ $\langle 0, 0, 1 \rangle$

$$\|v\| = \sqrt{3^2+5^2+1^2} = \sqrt{35}$$



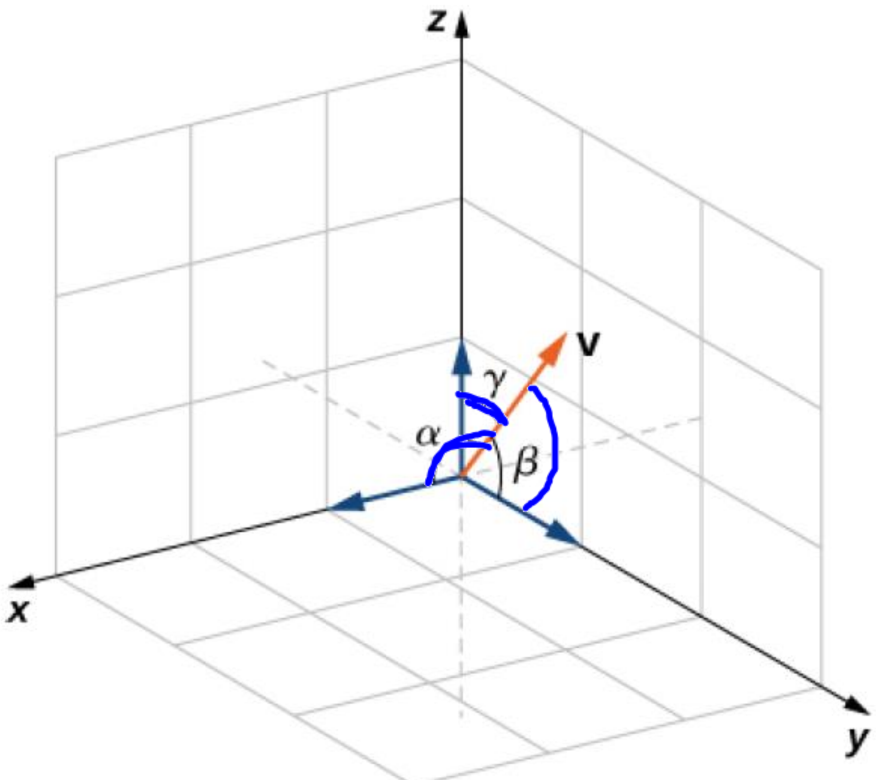
$$\cos\theta = \frac{v \cdot i}{\|v\|\,\|i\|} = \frac{3}{\sqrt{35}}$$

$$\cos\alpha = \frac{v \cdot j}{\|v\|\,\|j\|} = \frac{-5}{\sqrt{35}} = \frac{-\sqrt{35}}{7}$$

$$\cos\gamma = \frac{1}{\sqrt{35}}$$

Definition

The angles formed by a nonzero vector and the coordinate axes are called the **direction angles** for the vector (**Figure 2.48**). The cosines for these angles are called the **direction cosines**.



Definition

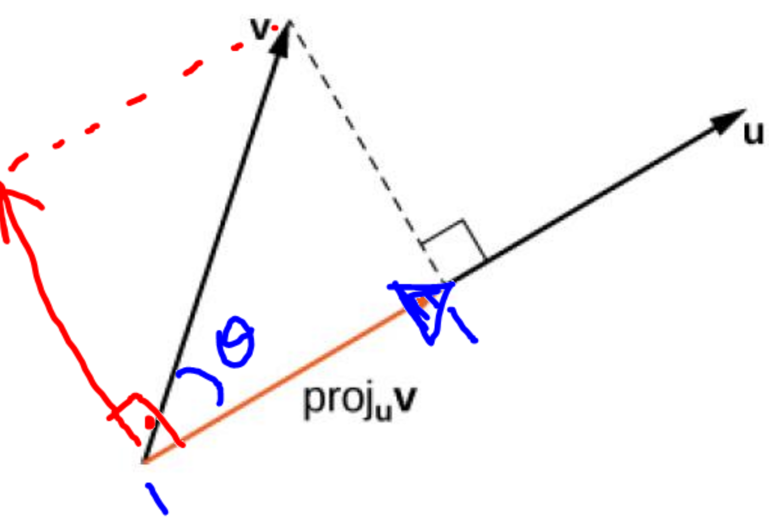
The **vector projection** of $\mathbf{v}$ onto $\mathbf{u}$ is the vector labeled $\text{proj}_{\mathbf{u}}\mathbf{v}$ in **Figure 2.50**. It has the same initial point as $\mathbf{u}$ and $\mathbf{v}$ and the same direction as $\mathbf{u}$, and represents the component of $\mathbf{v}$ that acts in the direction of $\mathbf{u}$. If θ represents the angle between $\mathbf{u}$ and $\mathbf{v}$, then, by properties of triangles, we know the length of $\text{proj}_{\mathbf{u}}\mathbf{v}$ is $\|\text{proj}_{\mathbf{u}}\mathbf{v}\| = \|\mathbf{v}\| \cos\theta$.

$$\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}\left(\underbrace{\frac{1}{\|\mathbf{u}\|}\mathbf{u}}_{\text{unit vector}}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\mathbf{u}. \tag{2.6}$$

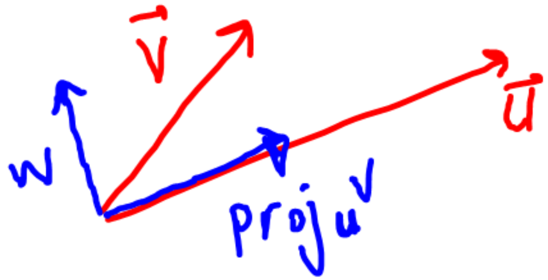
The length of this vector is also known as the **scalar projection** of $\mathbf{v}$ onto $\mathbf{u}$ and is denoted by

$$\|\text{proj}_{\mathbf{u}}\mathbf{v}\| = \text{comp}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|}. \tag{2.7}$$

$$\text{comp}_u v = \|v\| \cos\theta = \cancel{\|v\|}\,\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\cancel{\|\vec{v}\|}}$$



2.27 Express $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$ as a sum of orthogonal vectors such that one of the vectors has the same direction as $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j}$.



$$\text{proj}_u v = \frac{v \cdot u}{\|u\|^2} \cdot u = \frac{20-2}{4^2+2^2} \cdot (4i+2j)$$

$$= \frac{9}{10}(4i+2j) = 3.6i + 1.8j$$

$$w + \text{proj}_u v = v$$

$$w = v - \text{proj}_u v$$

$$w = (5i - j) - (3.6i + 1.8j)$$

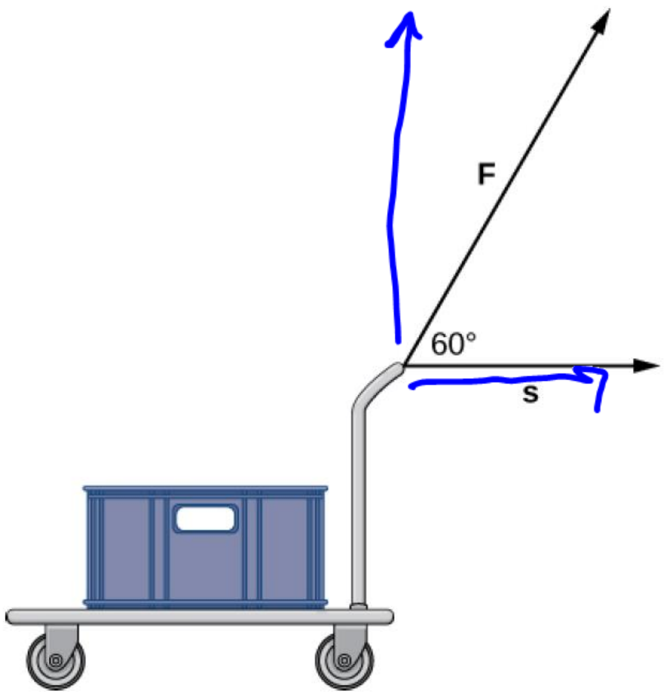
$$= 1.4i - 2.8j$$

Definition

When a constant force is applied to an object so the object moves in a straight line from point P to point Q, the work W done by the force $\mathbf{F}$, acting at an angle θ from the line of motion, is given by

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \| \mathbf{F} \| \ \| \overrightarrow{PQ} \| \cos \theta. \tag{2.8}$$

2.29 A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground (**Figure 2.52**). What is the work done by this force?



$W = 30 \times 10 \times \cos 60^{\circ}$

$= 150$ joule.

148. Determine all three-dimensional vectors **u** orthogonal to vector $\mathbf{v} = \mathbf{i} - \mathbf{j} - \mathbf{k}$. Express the answer in component form.

$v = \langle 1, -1, -1 \rangle$

$u = \langle x, y, z \rangle$

$u \cdot v = 0$

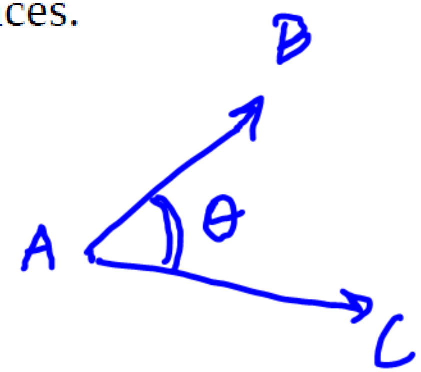
$x - y - z = 0$ a plane in 3-D.

$\langle 1, 1, 0 \rangle$

$\langle 1, 0, 1 \rangle$

$\langle 2, 1, 1 \rangle$

153. Determine the measure of angle A in triangle ABC, where $A(1,\ 1,\ 8)$, $B(4,\ -3,\ -4)$, and $C(-3,\ 1,\ 5)$. Express your answer in degrees rounded to two decimal places.



$$\vec{AB} = \langle 3, -4, -12 \rangle$$

$$\vec{AC} = \langle -4, 0, -3 \rangle$$

$$\cos\theta = \frac{\vec{AB} \cdot \vec{AC}}{\|AB\|\,\|AC\|}$$

$$= \frac{-12 + 0 + 36}{\sqrt{9+16+144}\,\sqrt{16+9}}$$

$$= \frac{24}{13 \times 5} = \frac{24}{65}$$

$$\theta = \arccos\left(\frac{24}{65}\right) = 68.33^\circ$$

Definition

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then, the **cross product** $\mathbf{u} \times \mathbf{v}$ is vector

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k} \\ &= \langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle . \end{aligned} \tag{2.9}$$

2.30 Find $\mathbf{p} \times \mathbf{q}$ for $\mathbf{p} = \langle 5, 1, 2 \rangle$ and $\mathbf{q} = \langle -2, 0, 1 \rangle$. Express the answer using standard unit vectors.

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$p \times q = \begin{vmatrix} i & j & k \\ 5 & 1 & 2 \\ -2 & 0 & 1 \end{vmatrix} = i(1-0) - j(5+4) + k(0+2)$$

$$= i - 9j + 2k$$

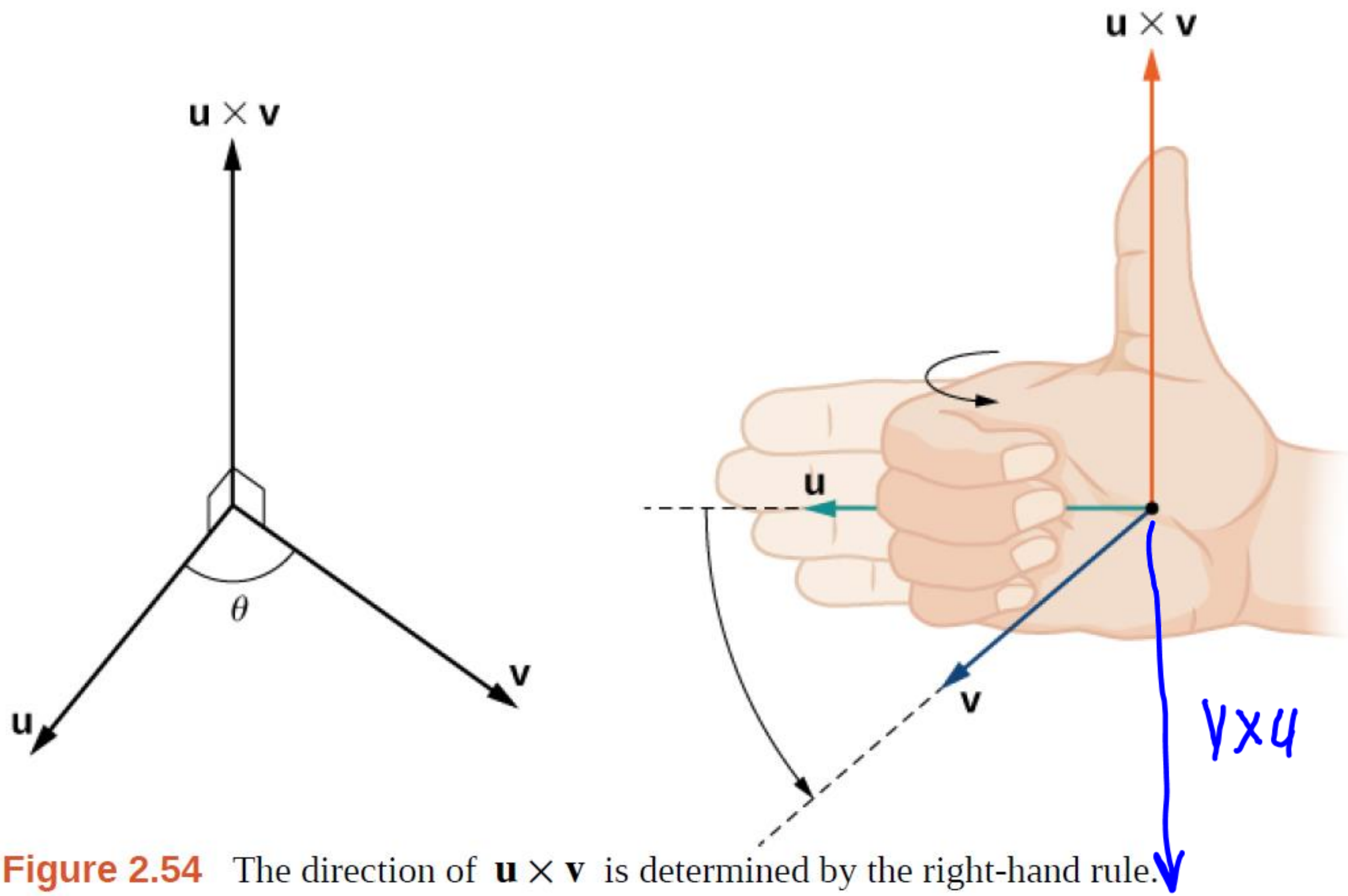
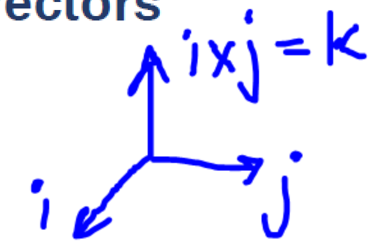
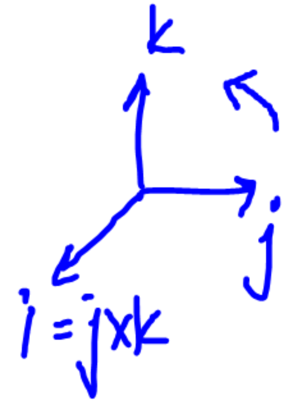


Figure 2.54 The direction of $\mathbf{u} \times \mathbf{v}$ is determined by the right-hand rule.

Cross Product of Standard Unit Vectors

$$\begin{aligned} \mathbf{i}\times\mathbf{j} &= \mathbf{k} & \mathbf{j}\times\mathbf{i} &= -\mathbf{k} \\ \mathbf{j}\times\mathbf{k} &= \mathbf{i} & \mathbf{k}\times\mathbf{j} &= -\mathbf{i} \\ \mathbf{k}\times\mathbf{i} &= \mathbf{j} & \mathbf{i}\times\mathbf{k} &= -\mathbf{j}. \end{aligned}$$





2.32 Find $(\mathbf{i}\times\mathbf{j})\times(\mathbf{k}\times\mathbf{i})$. $= k\times j = -i$

Theorem 2.6: Properties of the Cross Product

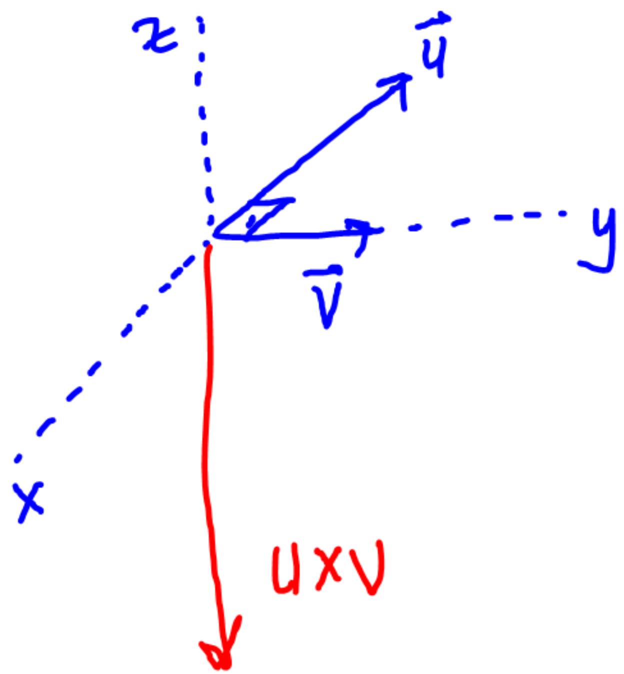
Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.

i.	$\mathbf{u}\times\mathbf{v} = -(\mathbf{v}\times\mathbf{u})$		Anticommutative property
ii.	$\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v}+\mathbf{u}\times\mathbf{w}$		Distributive property
iii.	$c(\mathbf{u}\times\mathbf{v}) = (c\mathbf{u})\times\mathbf{v} = \mathbf{u}\times(c\mathbf{v})$		Multiplication by a constant
iv.	$\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$		Cross product of the zero vector
v.	$\mathbf{v}\times\mathbf{v} = \mathbf{0}$	vector	Cross product of a vector with itself
vi.	$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$		Scalar triple product

Theorem 2.7: Magnitude of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors, and let θ be the angle between them. Then $\| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \cdot \| \mathbf{v} \| \cdot \sin\theta$.

2.34 Use **Properties of the Cross Product** to find the magnitude of $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = \langle -8, 0, 0 \rangle$ and $\mathbf{v} = \langle 0, 2, 0 \rangle$.



$$\|u \times v\| = \|u\| \|v\| \sin\theta$$
$$= 8 \times 2 \times \sin\frac{\pi}{2} = 16$$

$$u \times v = -16k = \langle 0, 0, -16 \rangle$$

Rule: Cross Product Calculated by a Determinant

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Then the cross product $\mathbf{u} \times \mathbf{v}$ is given by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}.$$

2.36 Use determinant notation to find $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a} = \langle 8, 2, 3 \rangle$ and $\mathbf{b} = \langle -1, 0, 4 \rangle$.

$$a \times b = \begin{vmatrix} i & j & k \\ 8 & 2 & 3 \\ -1 & 0 & 4 \end{vmatrix} = i\begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} - j\begin{vmatrix} 8 & 3 \\ -1 & 4 \end{vmatrix} + k\begin{vmatrix} 8 & 2 \\ -1 & 0 \end{vmatrix}$$

$$= i\,8 - j(32+3) + k(0-(-2))$$

$$= 8i - 35j + 2k$$

2.37 Find a unit vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a} = \langle 4, 0, 3 \rangle$ and $\mathbf{b} = \langle 1, 1, 4 \rangle$.

$$a \times b = \begin{vmatrix} i & j & k \\ 4 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix} = -3i - j\,13 + k\,4$$

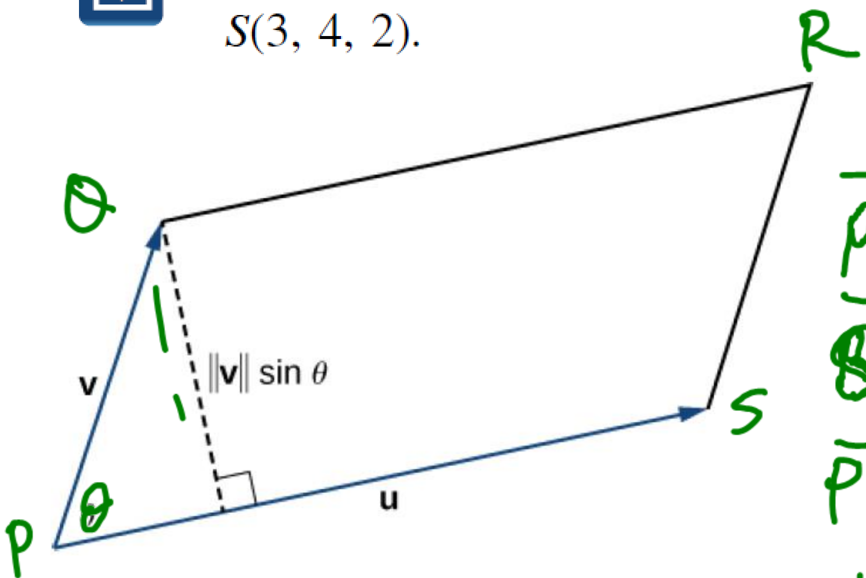
$$\frac{a \times b}{\| a \times b \|} = \frac{-3i - 13j + 4k}{\sqrt{3^2 + 13^2 + 4^2}} = \frac{\langle -3, -13, 4 \rangle}{\sqrt{194}}$$

a unit vector

Theorem 2.8: Area of a Parallelogram

If we locate vectors **u** and **v** such that they form adjacent sides of a parallelogram, then the area of the parallelogram is given by $\|\mathbf{u} \times \mathbf{v}\|$ (Figure 2.57).

2.38 Find the area of the parallelogram *PQRS* with vertices $P(1, 1, 0)$, $Q(7, 1, 0)$, $R(9, 4, 2)$, and $S(3, 4, 2)$.



$\|PQ \times PS\| = \text{Area}$

$\vec{PQ} = \langle 6,0,0\rangle$

$\vec{SR} = \langle 6,0,0\rangle$

$\vec{PS} = \langle 2,3,2\rangle$

$$PQ \times PS = \begin{vmatrix} i & j & k \\ 6 & 0 & 0 \\ 2 & 3 & 2 \end{vmatrix}$$

$$= -j\,12 + 18k$$

$$\|PQ \times PS\| = \sqrt{12^2 + 18^2}$$
$$= 6\sqrt{2^2 + 3^2}$$
$$= 6\sqrt{13} = \text{Area.}$$

$$A = \|u\|\,h = \|u\|\,\|v\| \sin\theta = \|u \times v\|$$

Definition

The **triple scalar product** of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

The triple scalar product of vectors $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$ is the determinant of the 3×3 matrix formed by the components of the vectors:

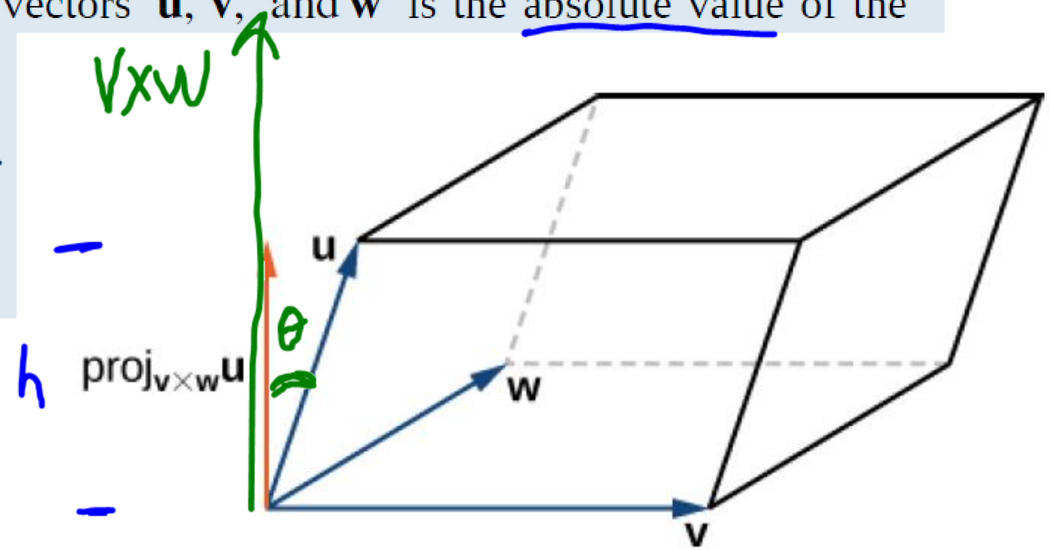
$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \quad = (u \times v) \cdot w$$

Theorem 2.10: Volume of a Parallelepiped

The volume of a parallelepiped with adjacent edges given by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is the absolute value of the triple scalar product:

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

See **Figure 2.59**.



$$= \|u\| \, \|v \times w\| \cos\theta$$

$$= A \quad h \quad = \text{Volume}.$$

2.39 Calculate the triple scalar product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, where $\mathbf{a} = \langle 2, -4, 1 \rangle$, $\mathbf{b} = \langle 0, 3, -1 \rangle$, and $\mathbf{c} = \langle 5, -3, 3 \rangle$.

$$\begin{vmatrix} 2 & -4 & 1 \\ 0 & 3 & -1 \\ 5 & -3 & 3 \end{vmatrix} = (-1)0\cdot\begin{vmatrix} -4 & 1 \\ -3 & 3 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & -4 \\ 5 & -3 \end{vmatrix}$$

$$= 0 + 3 + 14 = 17$$

$$\begin{array}{ccccc} 15 & & & & 18 \\ 6 & 2 & -4 & 1 & 0 \\ 0 & 0 & 3 & -1 & 20 \end{array}$$

$$38 - 21 = 17.$$

2.40 Find the volume of the parallelepiped formed by the vectors $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$, and $\mathbf{c} = 3\mathbf{j} + \mathbf{k}$.

$$V = a \cdot (b \times c) = \begin{vmatrix} 3 & 4 & -1 \\ 2 & -1 & -1 \\ 0 & 3 & 1 \end{vmatrix} = -9 - (-1) = \underline{-8}$$

$$\begin{array}{cccccc} 0 & & 3 & 4 & -1 & -3 \\ -9 & & & & & -6 \\ & & 2 & -1 & -1 & 0 \\ +8 & & & & & -9 \\ \hline -1 & & & & & \end{array}$$

volume is 8 unit^3.