3.3 arclength, curvature

review questions,

3.4 Motion in space

108. Find the arc length of the vector-valued function $\mathbf{r}(t) = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}$ over $\underline{[0, 1]}$. $0 \leq t \leq 1$

$r(t)$ is line segment from $(0,0,0)$ to $(-1,4,3)$

$$S = \int_0^1 \|r'(t)\|\,dt$$

$$S = \int_0^1 \sqrt{26}\,dt = \sqrt{26}\,t\Big|_0^1 = \sqrt{26}.$$

$$r'(t) = -1\,i + 4j + 3k$$

$$\|r'(t)\| = \sqrt{1^2 + 4^2 + 3^2} = \sqrt{26}$$

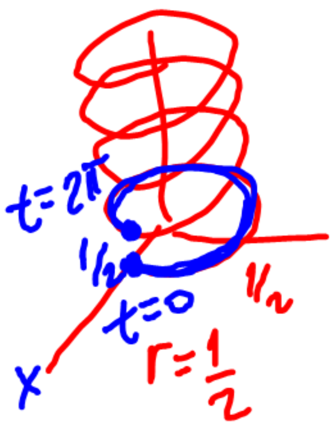
$$r(t) = \langle 0,0,0\rangle + t\langle -1, 4, 3\rangle$$

107. Find the length of one turn of the helix given by $\mathbf{r}(t) = \frac{1}{2}\cos t\,\mathbf{i} + \frac{1}{2}\sin t\,\mathbf{j} + \sqrt{\frac{3}{4}}\,t\,\mathbf{k}$.

$0 \leqslant t \leqslant 2\pi$

$r'(t) = -\frac{\sin t}{2} i + \frac{\cos t}{2} j + \sqrt{\frac{3}{4}} k$

$\|r'(t)\| = \frac{\sin^2 t + \cos^2 t}{4} + \frac{3}{4} = \frac{1+3}{4} = 1$



$s = \int_0^{2\pi} 1\, dt = t \Big|_0^{2\pi} = 2\pi$

$x^2 + y^2 = r^2$

$x = \frac{1}{2}\cos t$

$y = \frac{1}{2}\sin t$

$t=0, \quad r(0) = \frac{1}{2} i$

$t=2\pi \quad r(2\pi) = \frac{1}{2} i + 0j + \sqrt{\frac{3}{4}}\, 2\pi k$

115. Given $\mathbf{r}(t) = \langle 2e^t, e^t \cos t, e^t \sin t \rangle$, determine the tangent vector $\mathbf{T}(t)$.

unit

$$r'(t) = \langle 2e^t, e^t\cos t + e^t(-\sin t), e^t \sin t + e^t \cos t \rangle$$

$$= e^t \langle 2, \cos t - \sin t, \sin t + \cos t \rangle$$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

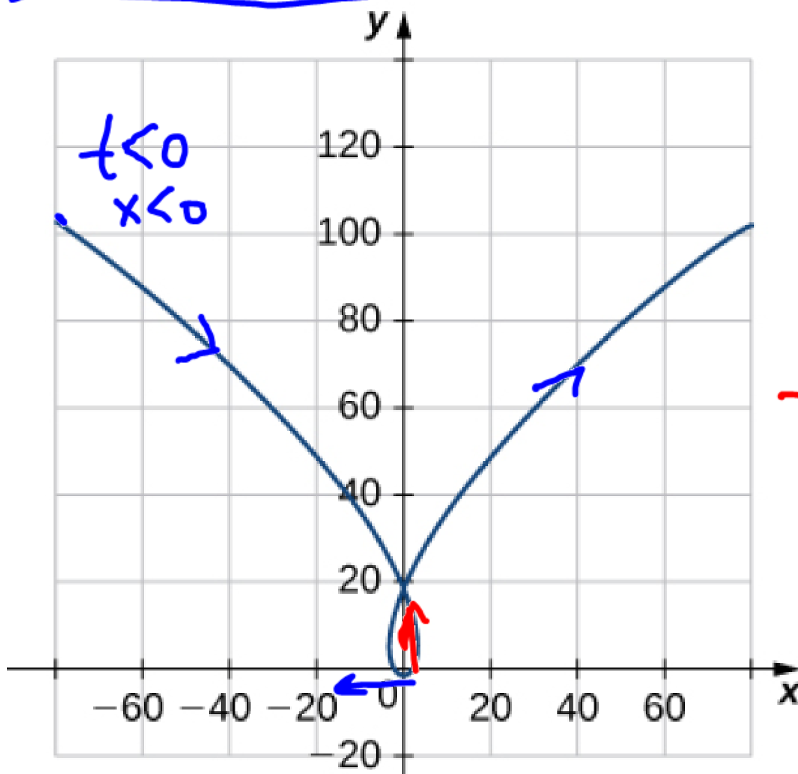
$$\|r'(t)\| = e^t \sqrt{2^2 + (\cos t - \sin t)^2 + (\sin t + \cos t)^2}$$

$$= e^t \sqrt{4 + \cos^2 t \cancel{- 2\cos t \sin t} + \sin^2 t + \sin^2 t \cancel{+ 2 \sin t \cos t} + \cos^2 t}$$

$$= e^t \sqrt{4+1+1}, \quad \sin^2 t + \cos^2 = 1$$

$$T(t) = \frac{\langle 2, \cos t - \sin t, \sin t + \cos t \rangle}{\sqrt{6}}$$

120. Find the unit tangent vector $\mathbf{T}(t)$ and unit normal vector $\mathbf{N}(t)$ at $t = 0$ for the plane curve $\mathbf{r}(t) = \langle t^3 - 4t, 5t^2 - 2 \rangle$. The graph is shown here:



$T = \langle -1, 0 \rangle = -i$

$$T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$$r'(t) = \langle 3t^2 - 4, 10t \rangle$$

$$\|r'(t)\| = \sqrt{(3t^2-4)^2 + (10t)^2}$$

$$T(0) = \frac{\langle -4, 0 \rangle}{4}$$

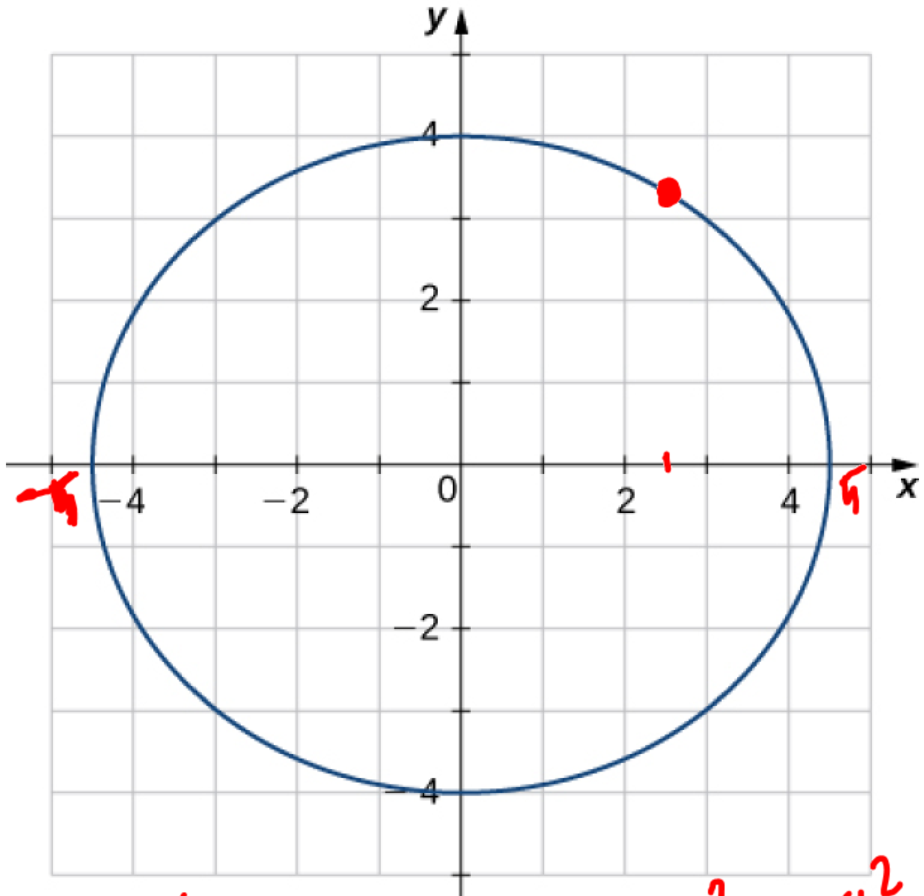
$$T(0) = \langle -1, 0 \rangle$$

$$T(t) = \left\langle \frac{3t^2 - 2}{\sqrt{(3t^2-4)^2 + (10t)^2}}, \frac{10t}{\sqrt{(3t-4)^2 + (10t)^2}} \right\rangle$$

Try to calulate N(0)

130. Find the ~~curvature~~ of the curve $\mathbf{r}(t) = 5\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j}$ at $t = \pi/3$. (*Note:* The graph is an ellipse.)

$$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{|y''|}{[1+(y')^2]^{3/2}}$$



$$r'(t) = \langle -5\sin t, 4\cos t\rangle$$

$$T(t) = \frac{\langle -5\sin t, 4\cos t\rangle}{\sqrt{25\sin^2 t + 16\cos^2 t}}$$

$$T'(t) = \langle -5\cos t\sqrt{\ldots} + 5\sin t$$

$50\sin t$

$$\cos t = \frac{x}{5} \qquad \sin t = \frac{y}{4} \qquad \frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$r(t) = \langle 5\cos t, 4\sin t, 0 \rangle$$

$$r'(t) = \langle -5\sin t, 4\cos t, 0 \rangle$$

$$r''(t) = \langle -5\cos t, -4\sin t, 0 \rangle$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -5\sin t & 4\cos t & 0 \\ -5\cos t & -4\sin t & 0 \end{vmatrix}$$

$$= i \cdot 0 - j \cdot 0 + k\left(20\sin^2 t + 20\cos^2 t\right)$$

$$K = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{20}{\left(\frac{\sqrt{91}}{2}\right)^3} = \frac{160}{91\sqrt{91}}$$

$$\|r'(t)\| = \sqrt{25\sin^2 t + 16\cos^2 t} = \sqrt{9\sin^2 t + 16} = \underline{\underline{\frac{\sqrt{91}}{2}}} \text{ at } t = \frac{\pi}{3}$$

3.4 | Motion in Space

Definition

Let $\mathbf{r}(t)$ be a twice-differentiable vector-valued function of the parameter t that represents the position of an object as a function of time. The **velocity vector** $\mathbf{v}(t)$ of the object is given by

$$\text{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t). \tag{3.20}$$

The **acceleration vector** $\mathbf{a}(t)$ is defined to be

$$\text{Acceleration} = \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t). \tag{3.21}$$

The *speed* is defined to be

$$\text{Speed} = v(t) = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \frac{ds}{dt}. \tag{3.22}$$

3.14 A particle moves in a path defined by the vector-valued function $\mathbf{r}(t) = \left(t^2 - 3t\right)\mathbf{i} + (2t - 4)\,\mathbf{j} + (t + 2)\mathbf{k}$, where t measures time in seconds and where distance is measured in feet. Find the velocity, acceleration, and speed as functions of time.

t

$\mathbf{v}(t) = r'(t) = (2t-3)i + 2j + 1k$ velocity

$a(t) = v'(t) = 2i$ acceleration.

$v(t) = \|\mathbf{v}(t)\| = \sqrt{(2t-3)^2 + 2^2 + 1^2} = \sqrt{(2t-3)^2 + 5}$ Speed

Theorem 3.7: The Plane of the Acceleration Vector

The acceleration vector $\mathbf{a}(t)$ of an object moving along a curve traced out by a twice-differentiable function $\mathbf{r}(t)$ lies in the plane formed by the unit tangent vector $\mathbf{T}(t)$ and the principal unit normal vector $\mathbf{N}(t)$ to C. Furthermore,

$$\mathbf{a}(t) = v'(t)\mathbf{T}(t) + [v(t)]^2 \kappa \mathbf{N}(t).$$

Here, $v(t)$ is the speed of the object and κ is the curvature of C traced out by $\mathbf{r}(t)$.

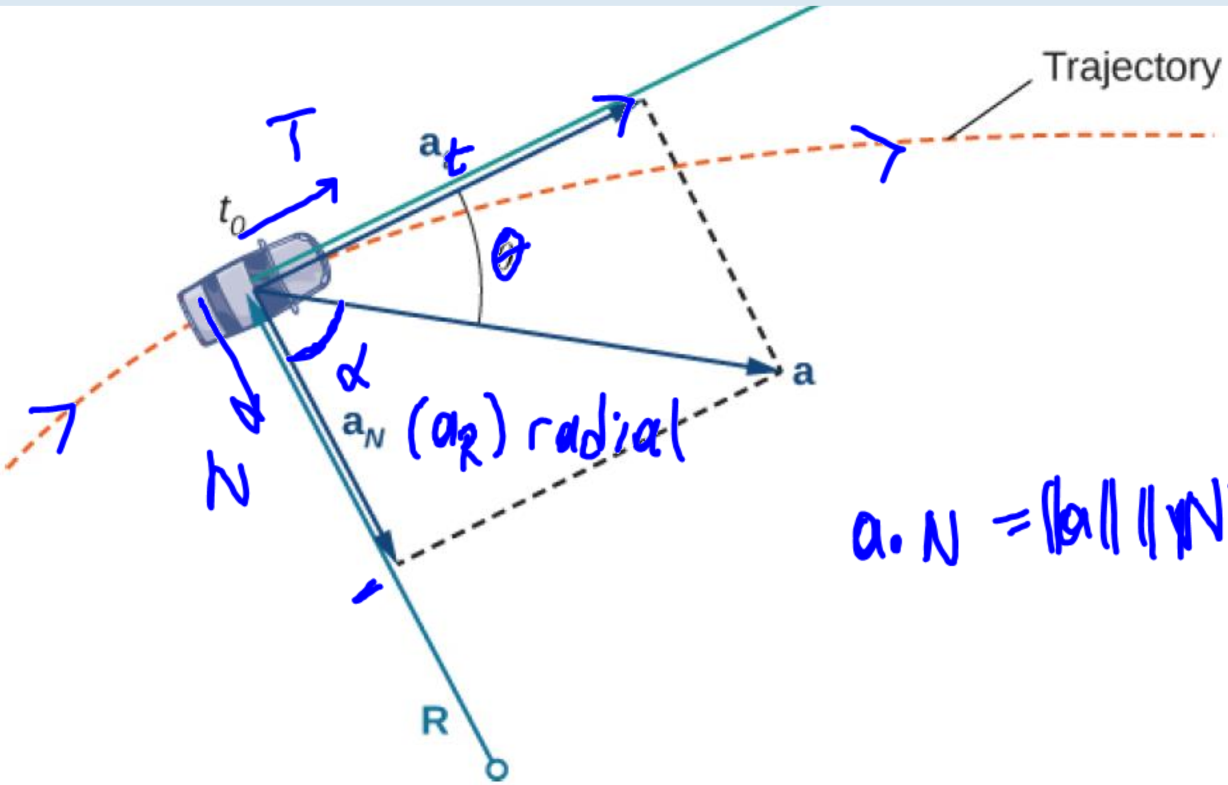


Figure 3.13 The dashed line represents the trajectory of an object (a car, for example). The acceleration vector points toward the inside of the turn at all times.

$v'(t)$

$\mathbf{T}(t)$: **tangential component of acceleration** $a_{\mathbf{T}}$

$\mathbf{N}(t)$ **normal component of acceleration** $a_{\mathbf{N}}$

$[v(t)]^2 \kappa$

$a \cdot N = \|a\| \, \|N\| \cos\alpha = \|a_N\|$

Theorem 3.8: Tangential and Normal Components of Acceleration

Let $\mathbf{r}(t)$ be a vector-valued function that denotes the position of an object as a function of time. Then $\mathbf{a}(t) = \mathbf{r}''(t)$ is the acceleration vector. The tangential and normal components of acceleration $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ are given by the formulas

$a \cdot T = \|a\| \|T\| \cos\theta$

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\| \mathbf{v} \|} \tag{3.23}$$

$T = \frac{r'(t)}{\|r'(t)\|} = \frac{v(t)}{\|v\|}$

and

$\|v \times a\| = \|v\| \|a\| \sin\theta$

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N} = \frac{\| \mathbf{v} \times \mathbf{a} \|}{\| \mathbf{v} \|} = \sqrt{\| \mathbf{a} \|^2 - a_{\mathbf{T}}^2}. \tag{3.24}$$

These components are related by the formula

$$\mathbf{a}(t) = a_{\mathbf{T}} \mathbf{T}(t) + a_{\mathbf{N}} \mathbf{N}(t). \tag{3.25}$$

Here $\mathbf{T}(t)$ is the unit tangent vector to the curve defined by $\mathbf{r}(t)$, and $\mathbf{N}(t)$ is the unit normal vector to the curve defined by $\mathbf{r}(t)$.

3.15 An object moves in a path defined by the vector-valued function $\mathbf{r}(t) = 4t\,\mathbf{i} + t^2\,\mathbf{j}$, where t measures time in seconds.

a. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ as functions of t.

b. Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at time $t = -3$.

parabola

$$V(t) = r'(t) = 4i + 2tj = \langle 4, 2t, 0\rangle$$

$$a(t) = r''(t) = 2j = \langle 0, t, 0\rangle$$

$$a_T = \frac{V \cdot a}{\|V\|} = \frac{4.0 + 2t.t + 0.0}{\sqrt{16+4t^2}} = \frac{2t^2}{\sqrt{16+4t^2}}$$

$$V \times a = \begin{vmatrix} i & j & k \\ 4 & 2t & 0 \\ 0 & t & 0 \end{vmatrix} = i.0 - j0 + k4t$$

$$a_N = \frac{\|V \times a\|}{\|V\|} = \frac{|4t|}{\sqrt{16+4t^2}}$$

$$a_T(-3) = \frac{18}{\sqrt{52}} = \frac{9}{\sqrt{13}}$$

$$a_N(-3) = \frac{12}{2\sqrt{13}} = \frac{6\sqrt{13}}{13}$$

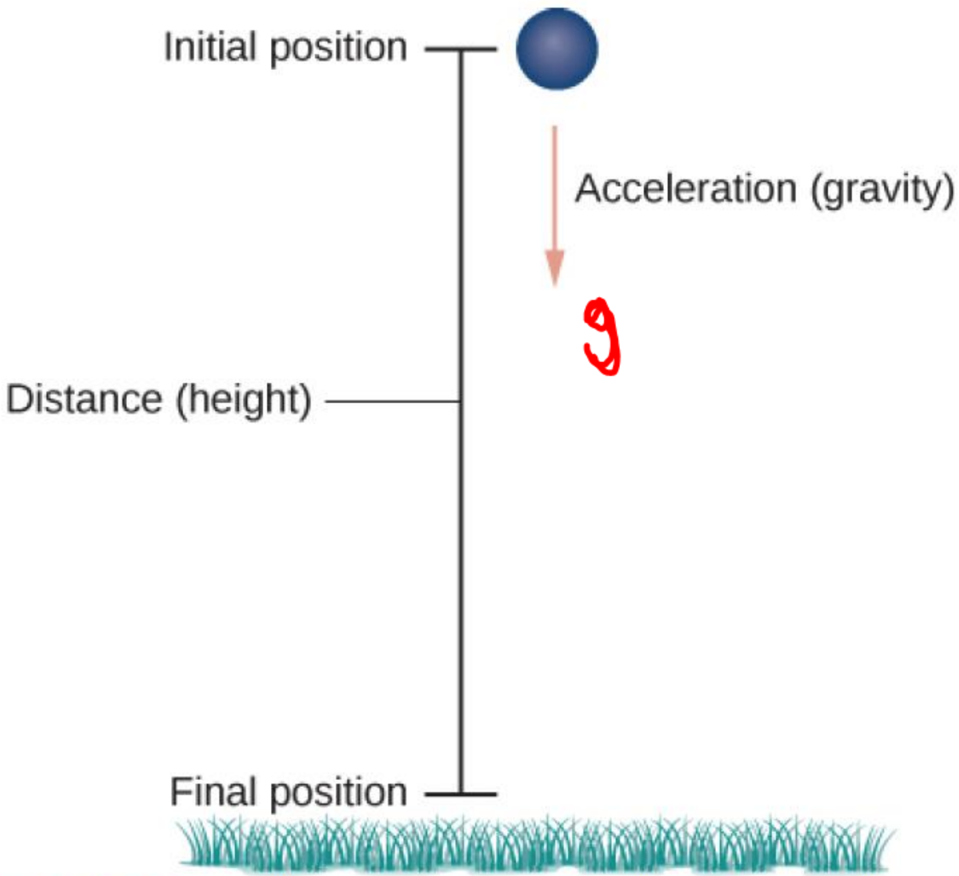


Figure 3.15 An object is falling under the influence of gravity.

$$F = ma = mg$$

$$a(t) = g$$

$$v(t) = gt + v_0 \quad , \quad v_0 = 0 \text{ for free-fall}$$

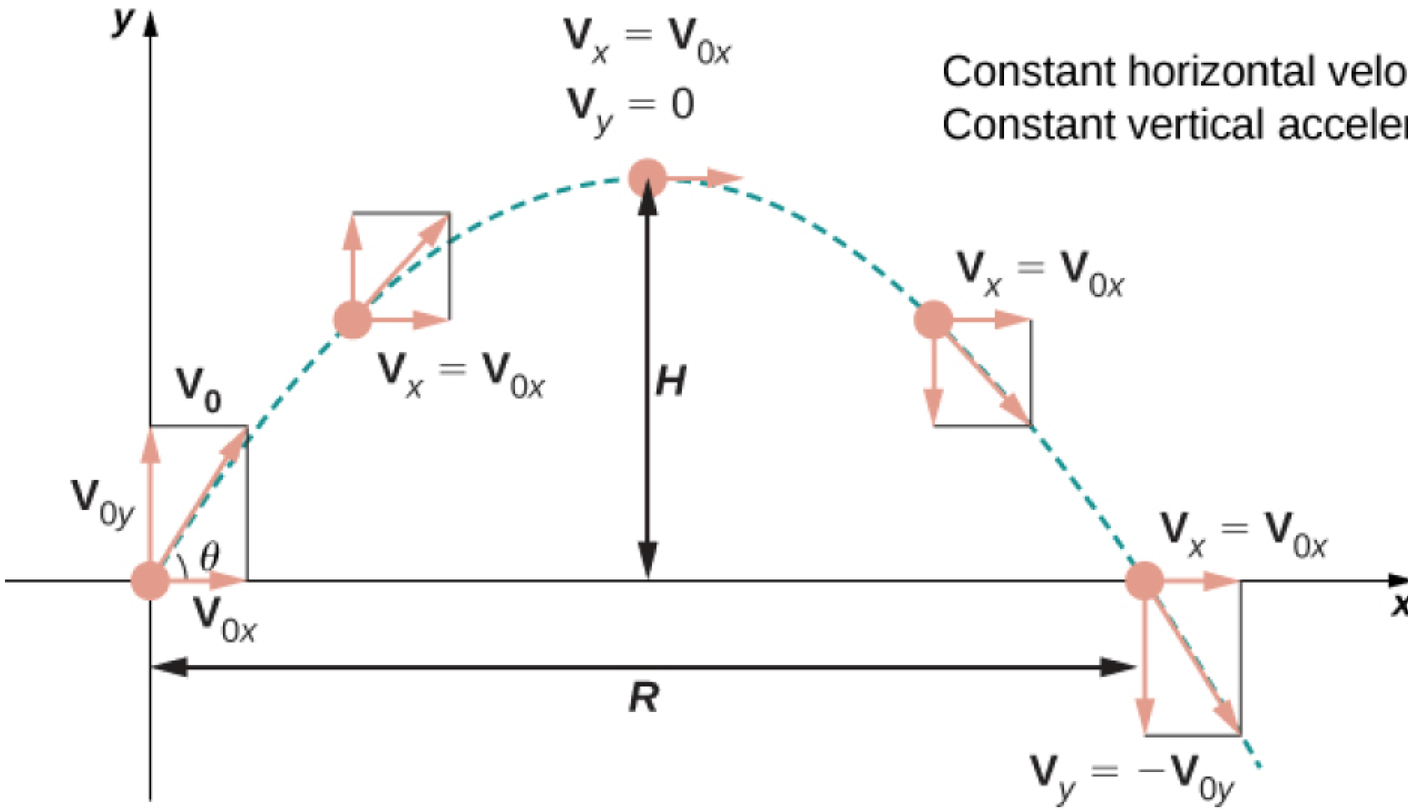
$$v(t) = gt$$

$$r(t) = \frac{1}{2} g t^2 + r_0 \quad , \quad r_0 = 0$$

$$r(t) = \frac{1}{2} g t^2$$

y-component of projectile motion

Projectile Motion



Constant horizontal velocity;
Constant vertical acceleration

$$\mathbf{v}_0 = v_0 \cos\theta \mathbf{i} + v_0 \sin\theta \mathbf{j}.$$

$$\boxed{\mathbf{s}(t) = \underline{-\tfrac{1}{2}gt^2}\,\mathbf{j} + \mathbf{v}_0 t + \mathbf{s}_0.}$$

$$\begin{aligned}
\mathbf{s}(t) &= -\tfrac{1}{2}gt^2\,\mathbf{j} + v_0 t\cos\theta\mathbf{i} + v_0 t\sin\theta\mathbf{j} \\
&= v_0 t\cos\theta\mathbf{i} + v_0 t\sin\theta\mathbf{j} - \tfrac{1}{2}gt^2\,\mathbf{j} \\
&= \underline{v_0 t\cos\theta}\mathbf{i} + \left(v_0 t\sin\theta - \underline{\tfrac{1}{2}gt^2}\right)\mathbf{j}.
\end{aligned}$$

Figure 3.16 Projectile motion when the object is thrown upward at an angle θ. The horizontal motion is at constant velocity and the vertical motion is at constant acceleration.

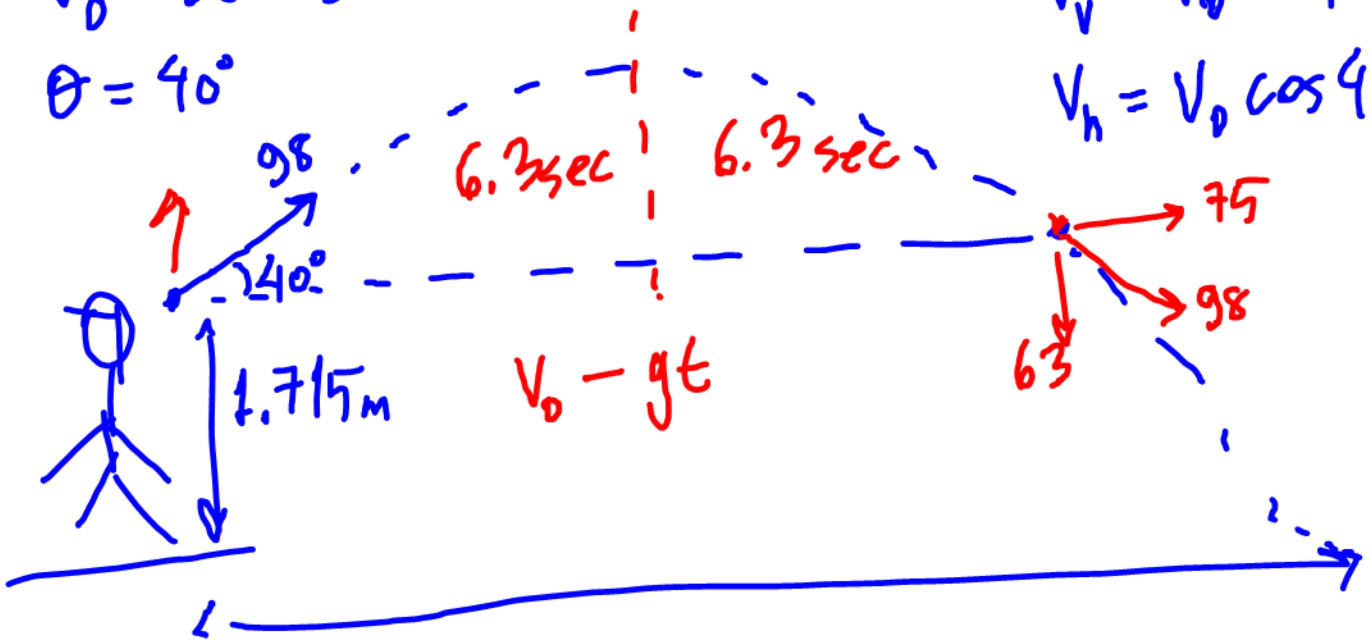
3.16 An archer fires an arrow at an angle of 40° above the horizontal with an initial speed of 98 m/sec. The height of the archer is 171.5 cm. Find the horizontal distance the arrow travels before it hits the ground.

$V_0 = 98$ m/s

$\theta = 40°$

$V_v = V_0 \sin 40° = 98 \cdot \sin 40° \approx 63$ m/s

$V_h = V_0 \cos 40° = 98 \cdot \cos 40° \approx 75$ m/s.



$1.71 = 63t + \frac{1}{2} g t^2$

$5t^2 + 63t - 1.715 = 0$