4.4 | Tangent Planes and Linear Approximations

Definition

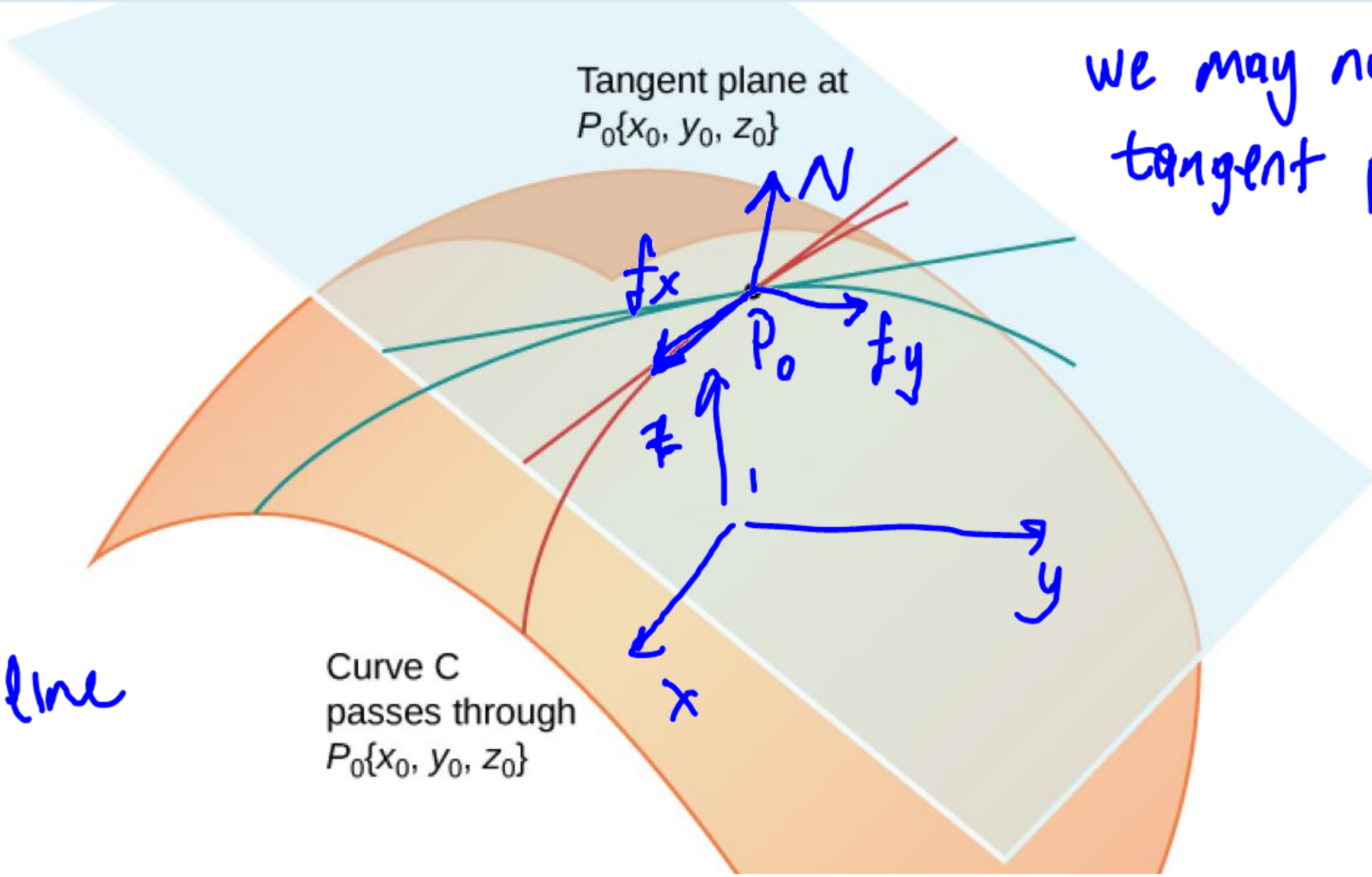
Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S, and let C be any curve passing through P_0 and lying entirely in S. If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the **tangent plane** to S at P_0 (**Figure 4.27**).

Calulus 1

$m_T = f'(x_0)$
slope of the tangent line

$|x|$

at $x=0$
no tangent line



we may not have tangent planes at some points.

Definition

Let S be a surface defined by a differentiable function $z = f(x, y)$, and let $P_0 = (x_0, y_0)$ be a point in the domain of f. Then, the equation of the tangent plane to S at P_0 is given by

$$\frac{\Delta z}{\Delta x} \qquad \boxed{z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).} \tag{4.24}$$

calculus 1

$f_x = \frac{\partial z}{\partial x}$ $\qquad$ $f_y = \frac{\partial z}{\partial y}$ $\qquad$ $y - y_0 = f'(x_0)(x - x_0)$

$y = y_0 + f'(x_0)(x - x_0)$

$u = \langle 1, 0, f_x \rangle$ $\quad$ $v = \langle 0, 1, f_y \rangle$

x constant

$N = u \times v$ is a normal vector for the tangent plane.

4.19 Find the equation of the tangent plane to the surface defined by the function $f(x, y) = x^3 - x^2 y + y^2 - 2x + 3y - 2$ at point $(-1, 3)$.

$$f(-1,3) = -1 - 3 + 9 + 2 + 9 - 2 = 14 = z_0 = f(x_0, y_0)$$

$$f_x = 3x^2 - 2xy - 2$$

$$f_x(-1,3) = 3 + 6 - 2 = 7$$

$$f_y = -x^2 + 2y + 3$$

$$f_y(-1,3) = -1 + 6 + 3 = 8$$

$$z = 14 + 7(x+1) + 8(y-3)$$

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Linear Approximations

Calculus 1

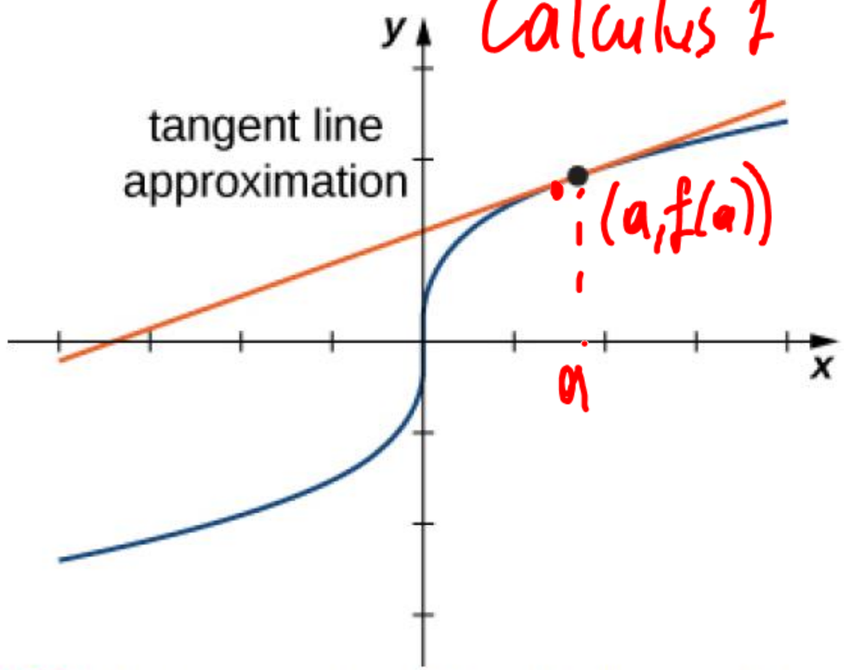


Figure 4.30 Linear approximation of a function in one variable.

$$y \approx f(a) + f'(a)(x - a).$$

tangent line $y - f(a) = f'(a)(x-a)$

$\sqrt{10}$ $\quad f'(x) = \frac{1}{2\sqrt{x}}$

$f(x) = \sqrt{x}$

$$\sqrt{10} \approx \sqrt{9} + \frac{1}{2\sqrt{9}}(10-9)$$

$$\approx 3 + \frac{1}{6} = \frac{19}{6}.$$

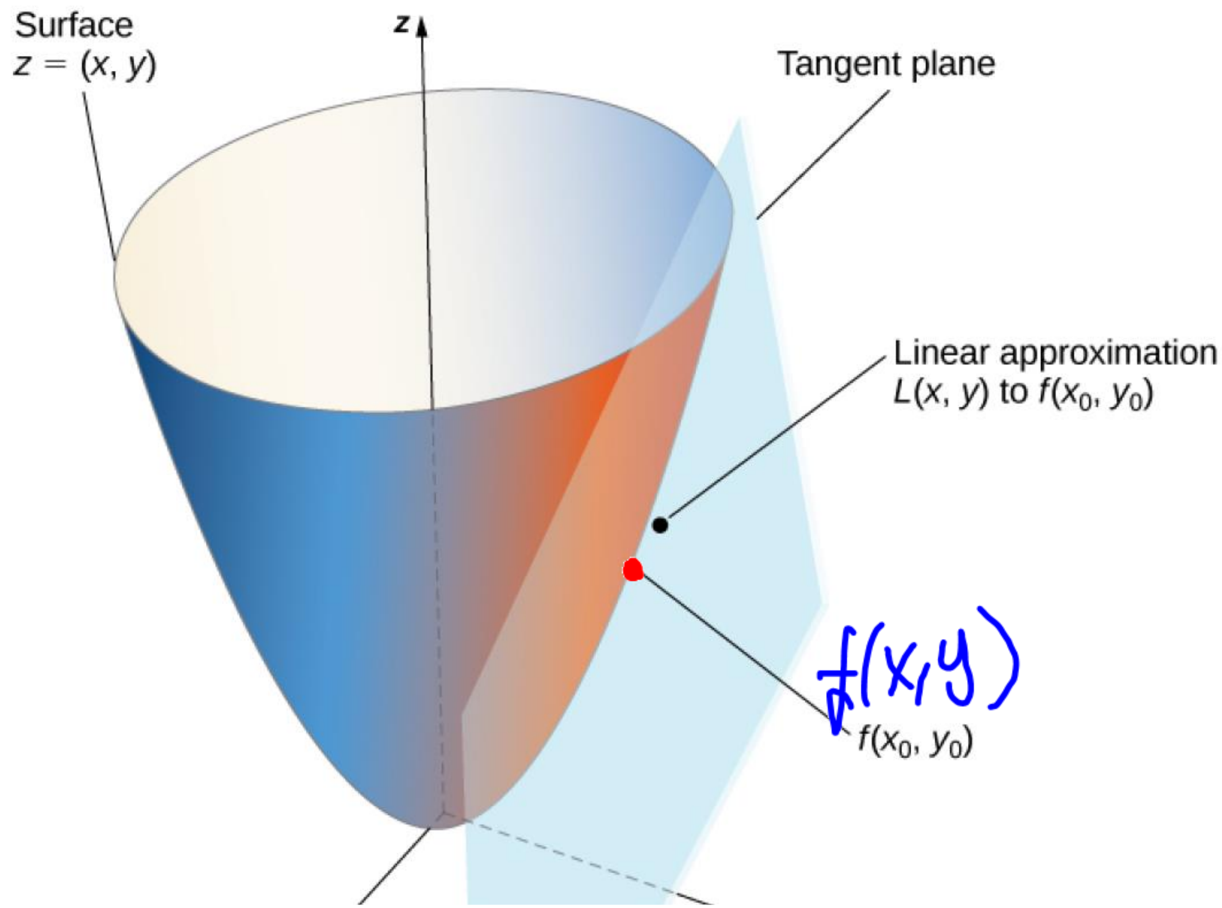
$\sqrt{10} = 3.162277\ldots$

$\frac{19}{6} = 3.166666\ldots$

Definition

Given a function $z = f(x, y)$ with continuous partial derivatives that exist at the point (x_0, y_0), the **linear approximation** of f at the point (x_0, y_0) is given by the equation

$$f(x,y) \approx L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \tag{4.25}$$



4.20 Given the function $f(x, y) = e^{5-2x+3y}$, approximate $\boxed{f(4.1, 0.9)}$ using point $(4, 1)$ for (x_0, y_0).

$x_0 \quad y_0$

What is the approximate value of $f(4.1, 0.9)$ to four decimal places?

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

$f(4,1) = e^{5-8+3} = e^0 = 1$

$f_x = -2e^{5-2x+3y}$

$f_x(4,1) = -2$

$f_y = 3e^{5-2x+3y}$

$f_y(4,1) = 3$

$L(x,y) = 1 + (-2)(x-4) + 3(y-1)$

$L(4.1, 0.9) = 1 - 2(0.1) + 3(-0.1)$

$= 1 - 0.2 - 0.3 = 0.5 \approx f(4.1, 0.9)$

exact value

$f(4.1, 0.9) = e^{5-8.2+2.7} = e^{7.7-8.2} = e^{-0.5} = 0.60\ldots$

For the following exercises, find the equation for the tangent plane to the surface at the indicated point. (*Hint:* Solve for z in terms of x and y.)

171. $z = -9x^2 - 3y^2$, $P(2, 1, -39)$

$x_0, y_0,$

$z = f(x_0, y_0) + f_x \ldots$

$f_x = -18x$

$f_y = -6y$

$z = -39 + (-36)(x-2) + (-6)(y-1)$

Differentiability

Definition

A function $f(x, y)$ is **differentiable** at a point $P(x_0, y_0)$ if, for all points (x, y) in a δ disk around P, we can write

$$f(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + E(x, y), \tag{4.26}$$

where the error term E satisfies

$$\lim_{(x, y) \to (x_0, y_0)} \frac{E(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0.$$

no tangent plane means not differentiable.

4.21 Show that the function $f(x, y) = 3x - 4y^2$ is differentiable at point $(-1, \overset{y_0}{2})$.

$$f(x,y) = L(x,y) + E(x,y)$$

$$f(-1,2) = -3 - 16 = -19$$

$$f_x = 3$$

$$f_y = -8y$$

$$L(x,y) = -19 + 3(x+1) - 16(y-2)$$

$$= 16 + 3x - 16y$$

$$E(x,y) = \cancel{3x} - 4y^2 - 16 - \cancel{3x} + 16y$$

$$y = 2$$

$$\lim_{(x,y)\to(-1,2)} \frac{-4y^2 - 16 + 16y}{\sqrt{(x+1)^2 + (y-2)^2}} = \lim_{x \to -1} \frac{-16 - 16 + 32}{\sqrt{(x+1)^2}} = 0.$$

Theorem 4.6: Differentiability Implies Continuity

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x, y)$ is differentiable at (x_0, y_0), then $f(x, y)$ is continuous at (x_0, y_0).

Theorem 4.7: Continuity of First Partials Implies Differentiability

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f. If $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ all exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0), then $f(x, y)$ is differentiable there.

Differentials

$dx \quad dy$

Definition

Let $z = f(x, y)$ be a function of two variables with (x_0, y_0) in the domain of f, and let Δx and Δy be chosen so that $(x_0 + \Delta x, y_0 + \Delta y)$ is also in the domain of f. If f is differentiable at the point (x_0, y_0), then the differentials dx and dy are defined as

$$dx = \Delta x \text{ and } dy = \Delta y.$$

The differential dz, also called the **total differential** of $z = f(x, y)$ at (x_0, y_0), is defined as

$$dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy. \tag{4.27}$$

dx is not ∂x

partial

4.22 Find the differential dz of the function $f(x, y) = 4y^2 + x^2 y - 2xy$ and use it to approximate Δz at point $(1, -1)$. Use $\Delta x = 0.03$ and $\Delta y = -0.02$. What is the exact value of Δz?

$$\boxed{dz = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy}$$

$$= (2xy - 2y)dx + (8y + x^2 - 2x)dy$$

$$= (-2+2)(0.03) + (-8+1-2)(-0.02)$$

$$= (-9)(-0.02)$$

$$= \underline{0.18} \text{ approximation.}$$

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

$$= f(1.03, -1.02) - f(1, -1)$$

$$= 5.180682\ldots - (4 - 1 + 2)$$

$$= \underline{0.180682}$$

exact value

Calculus 1

$$\frac{dy}{dx} = y'$$

$$\underline{dy = y'dx}$$

approximation

For the following exercises, as a useful review for techniques used in this section, find a normal vector and a tangent vector at point P.

165. $x^2 + xy + y^2 = 3$, $P(-1, -1)$

$(-1,-1,0)$

$z = x^2 + xy + y^2 - 3$

normal vector

$$N = \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{vmatrix}$$

$\langle 1, 0, f_x(-1,-1)\rangle = \langle 1, 0, -3\rangle$

$\langle 0, 1, f_y(-1,-1)\rangle = \langle 0, 1, -3\rangle$

are tangent vectors

$f_x = 2x + y$

$f_y = x + 2y$

$= 3i - j(-3) + k$

$= 3i + 3j + k = \langle 3, 3, 1\rangle$

OR

$z = 0 + (-3)(x+1) + (-3)(y+1)$ tangent plane

$z + 3x + 3y + 6 = 0$ $\quad N = \langle 3, 3, 1\rangle$

For the following exercises, find the equation for the tangent plane to the surface at the indicated point. (*Hint:* Solve for z in terms of x and y.)

174. $z = e^{7x^2 + 4y^2}$, $P(0, 0, 1)$

$f_x = 14x\, e^{7x^2+4y^2}$

$f_y = 8y\, e^{7x^2+4y^2}$

$z = 1 + 0(x-0) + 0(y-0)$

$z = 1$ is the equation of the tangent plane.

For the following exercises, find parametric equations for the normal line to the surface at the indicated point. (Recall that to find the equation of a line in space, you need a point on the line, $P_0(x_0, y_0, z_0)$, and a vector $\mathbf{n} = \langle a, b, c \rangle$ that is parallel to the line. Then the equation of the line is $x - x_0 = at,\ y - y_0 = bt,\ z - z_0 = ct$.)

183. $z = 5x^2 - 2y^2$, $P(2, 1, 18)$

tangent plane

$z = 18 + 20(x-2) - 4(y-1)$

$f_x = 10x$ $\quad N = \langle 20, -4, -1 \rangle$

$f_y = -4y$

$\boxed{0 = 20x - 4y - z - 18}$

$\langle 2, 1, 18 \rangle + t\langle 20, -4, -1 \rangle$

$$\left.\begin{aligned} x &= 2 + 20t \\ y &= 1 - 4t \\ z &= 18 - t \end{aligned}\right\}$$ parametric form of the normal line.

196. The volume of a right circular cylinder is given by $V(r, h) = \pi r^2 h$. Find the differential dV. Interpret the formula geometrically.

197. See the preceding problem. Use differentials to estimate the amount of aluminum in an enclosed aluminum can with diameter $8.0\,\text{cm}$ and height $12\,\text{cm}$ if the aluminum is 0.04 cm thick.

For the following exercises, find the linear approximation of each function at the indicated point.

208. $f(x, y) = e^x \cos y$; $P(0, 0)$

For the following exercises, find the linear approximation of each function at the indicated point.

207. $f(x, y) = x\sqrt{y}, \quad P(1, 4)$