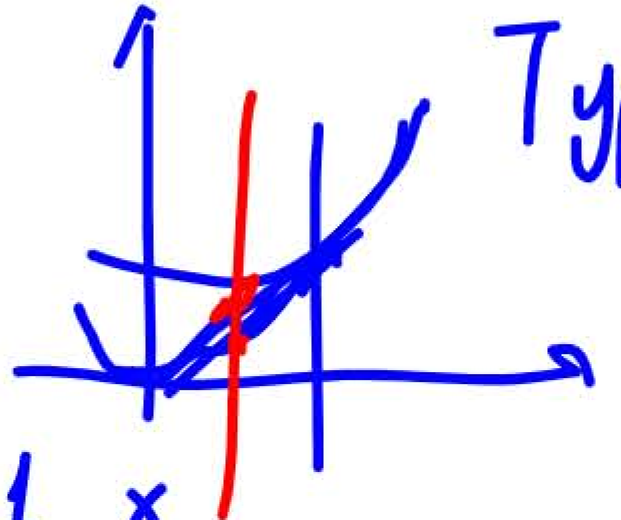
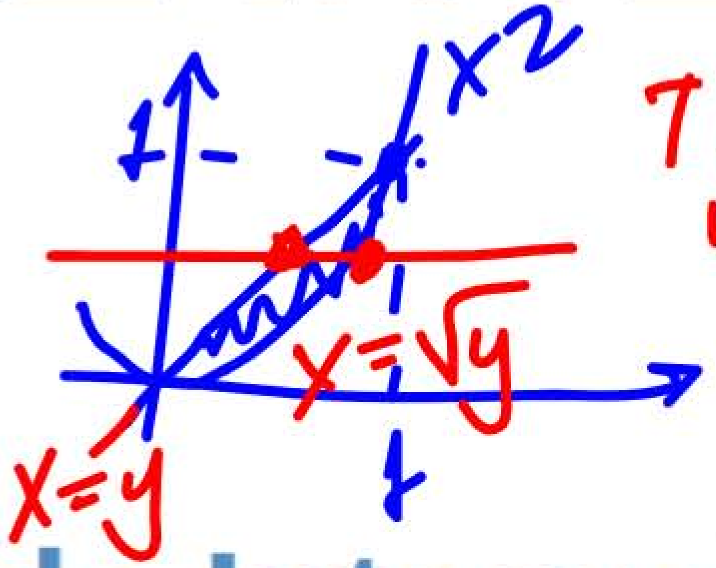
5 | MULTIPLE INTEGRATION



Type I $0 \le x \le 1$

$x^2 \le y \le x$

$$\int_0^1 \int_{x^2}^{x} \dots dy\,dx$$



Type II $0 \le y \le 1$

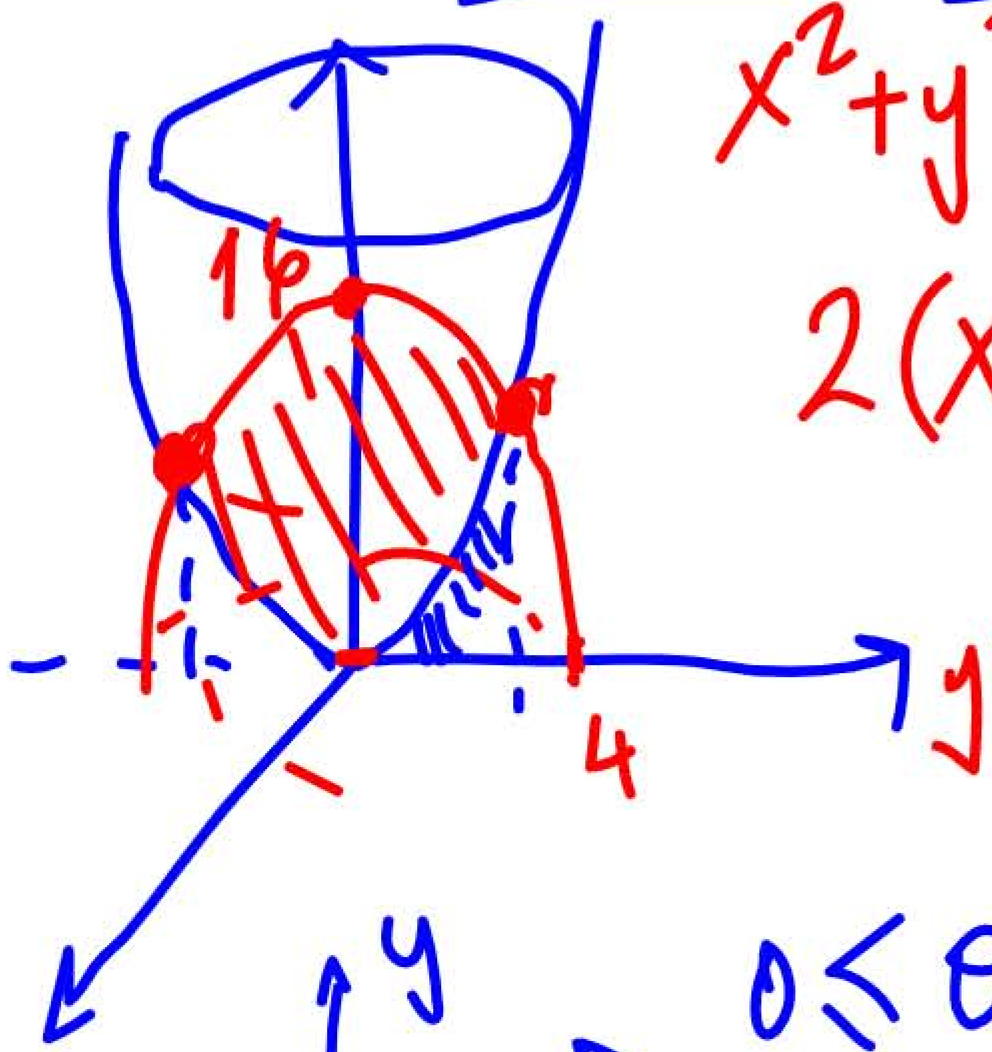
$y \le x \le \sqrt{y}$

$$\int_0^1 \int_{y}^{\sqrt{y}} \dots dx\,dy$$

5.4 | Triple Integrals

5.3 Double integrals in polar coordinates

5.20 Use polar coordinates to find an iterated integral for finding the volume of the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 16 - x^2 - y^2$.



$$x^2+y^2 = 16-(x^2+y^2)$$

$$2(x^2+y^2) = 16$$

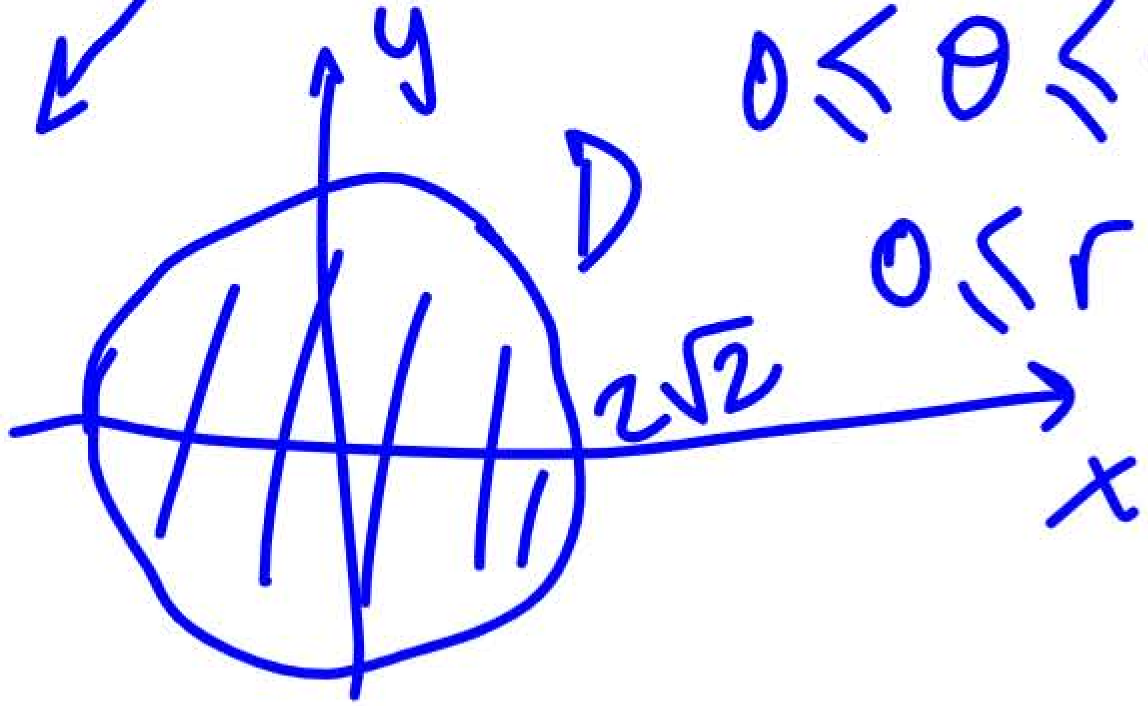
$$x^2+y^2 = 8 = (2\sqrt{2})^2$$

above below

$$V = \iint_D \left(16-x^2-y^2-(x^2+y^2)\right) dA$$

$$dA = r\,dr\,d\theta$$

in polar coordinates



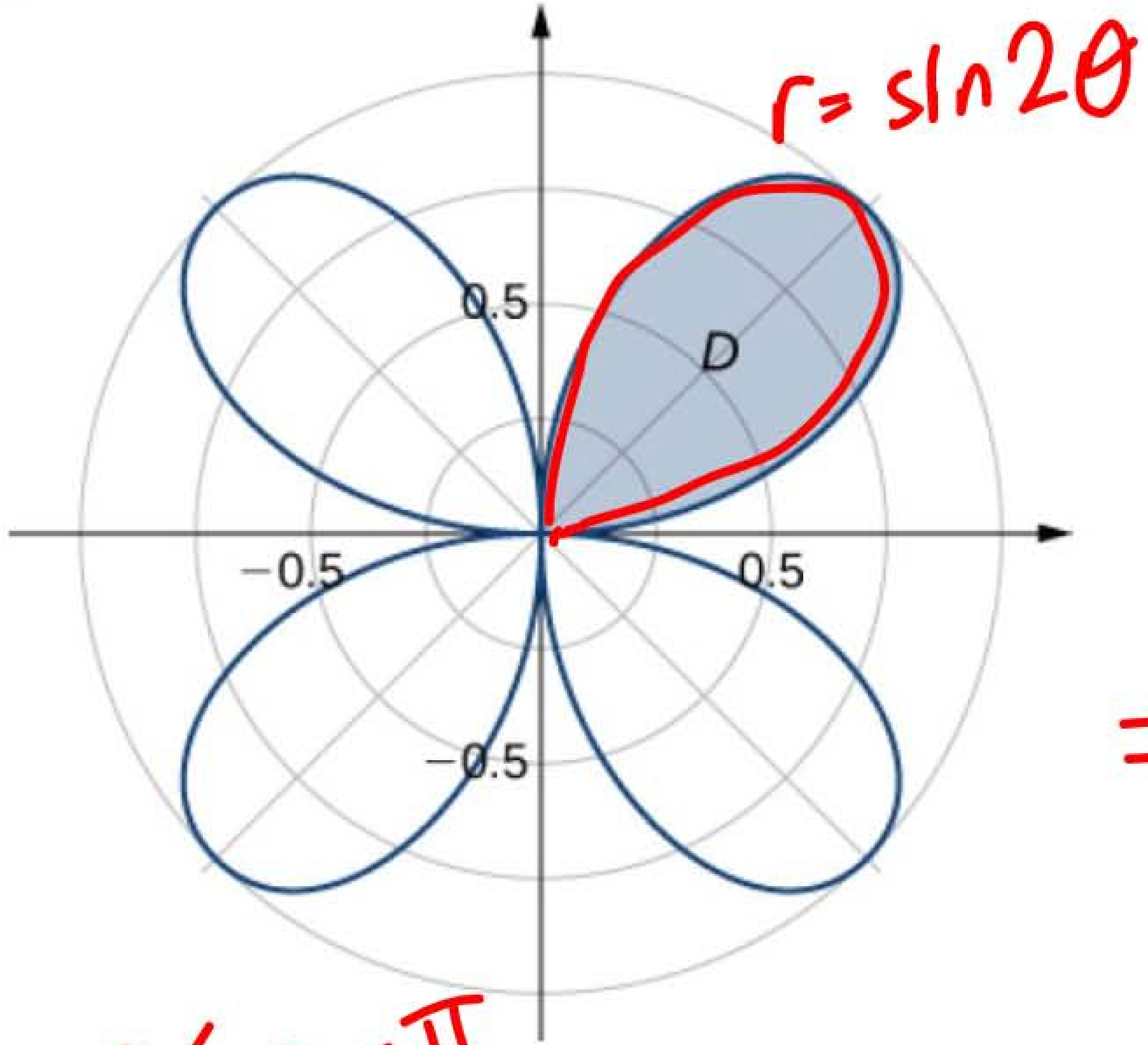
$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 2\sqrt{2}$$

$$\int_0^{2\pi}\int_0^{2\sqrt{2}} (16-r^2)\, r\,dr\,d\theta$$

$$\int_0^{2\pi} \left(8r^2 - \frac{r^4}{4}\right)\Bigg|_0^{2\sqrt{2}} d\theta = \int_0^{2\pi} 48\, d\theta = 96\pi$$

154. Evaluate the integral $\iint_D r\,dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see the following figure).



$0 \leqslant \theta \leqslant \pi/2$

$0 \leqslant r \leqslant \sin 2\theta$

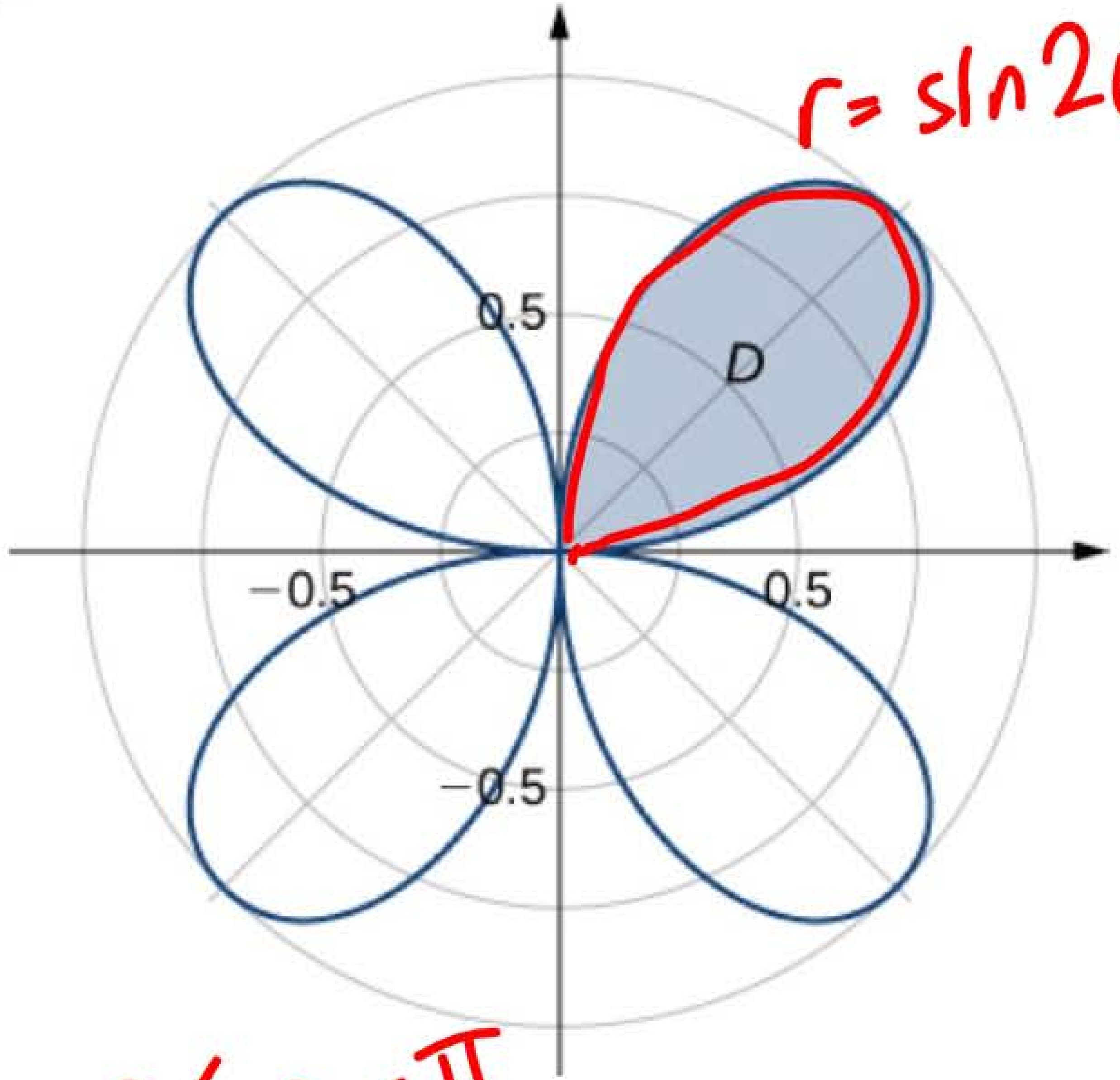
$$\int_0^{\pi/2}\int_0^{\sin 2\theta} r \cdot r\,dr\,d\theta$$

$$\int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{\sin 2\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^3 2\theta}{3}\,d\theta = \frac{1}{3}\int_0^{\pi/2} (1-\cos^2 2\theta)\sin 2\theta\,d\theta$$

$$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx \qquad u = \cos x \quad du = -\sin x\,dx$$

154. Evaluate the integral $\iint_D r\,dA$, where D is the region bounded by the part of the four-leaved rose $r = \sin 2\theta$ situated in the first quadrant (see the following figure).



$r = \sin 2\theta$

$0 \leq \theta \leq \pi/2$

$0 \leq r \leq \sin 2\theta$

$$\int_0^{\pi/2}\int_0^{\sin 2\theta} r \cdot r\,dr\,d\theta$$

$$\int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{\sin 2\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\sin^3 2\theta}{3}\,d\theta = \frac{1}{3}\int_0^{\pi/2} (1-\cos^2 2\theta)\sin 2\theta\,d\theta$$

$$\int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx$$

$u = \cos x$

$du = -\sin x\,dx$

$$\frac{1}{3}\int_0^{\pi/2}(1-\cos^2 2\theta)\sin 2\theta\, d\theta = \frac{1}{3}\int_1^{-1}(1-u^2)\left(-\frac{du}{2}\right)$$

$u = \cos 2\theta$

$du = -\sin 2\theta \cdot 2\, d\theta$

$\theta = 0 \quad u = \cos 0 = 1$

$\theta = \frac{\pi}{2} \quad u = \cos \pi = -1$

$$= \frac{1}{3}\left(\frac{u^3}{3} - u\right)\Big|_1^{-1}$$

$$= \frac{1}{3}\left(-\frac{1}{3} - (-1) - \left(\frac{1}{3} - 1\right)\right)$$

$$= \frac{1}{3}\,\frac{4}{3} = \frac{4}{9}$$

Techiques of Integr.

Calculus II

Slide 5

Definition

The **triple integral** of a function $f(x, y, z)$ over a rectangular box B is defined as

$$\lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta x \Delta y \Delta z = \iiint_B f(x, y, z) dV \tag{5.10}$$

if this limit exists.

$dV = dx\,dy\,dz$

$a \leq x \leq b$

$c \leq y \leq d$

$e \leq z \leq f$

Theorem 5.9: Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a rectangular box $B = [a, b] \times [c, d] \times [e, f]$, then

$$\iiint_B f(x, y, z) dV = \int_e^f \int_c^d \int_a^b f(x, y, z) dx\, dy\, dz.$$

This integral is also equal to any of the other five possible orderings for the iterated triple integral.

$$\int_e^f \int_a^b \int_c^d f(x, y, z)\, dy\, dx\, dz$$

Slide 6

In the following exercises, evaluate the triple integrals over the rectangular solid box B.

181. $\iiint\limits_B \left(2x + 3y^2 + 4z^3\right) dV,$ where

$B = \{(x, y, z) | 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3\}$

$$\int_0^3 \int_0^2 \boxed{\int_0^1 (2x+3y^2+4z^3)\,dx}\,dy\,dz = \int_0^3 \int_0^2 \left(x^2+(3y^2+4z^3)x\right)\Big|_0^1 dy\,dz$$

$$= \int_0^3 \underbrace{\int_0^2 (1+3y^2+4z^3)\,dy}_{\text{a double integral}}\,dz = \int_0^3 \left((1+4z^3)y + y^3\right)\Big|_0^2 dz$$

$$= \int_0^3 (2+8z^3+8)\,dz = (10z+2z^4)\Big|_0^3 = 192$$

5.23 Evaluate the triple integral $\iiint\limits_B z \sin x \cos y \, dV$ where $B = \left\{(x, y, z) \middle| 0 \le x \le \pi, \frac{3\pi}{2} \le y \le 2\pi, 1 \le z \le 3\right\}$.

rectangular box

$$\underbrace{\int_{3\pi/2}^{2\pi}\int_0^{\pi}\int_1^3 z \sin x \cos y \, dz}\, dx\, dy = \int_{3\pi/2}^{2\pi}\int_0^{\pi} \frac{z^2}{2} \sin x \cos y \Big|_1^3 \, dx\, dy$$

$$= \int_{3\pi/2}^{2\pi} 4(-\cos x)\cos y \Big|_0^{\pi} dy = \int_{3\pi/2}^{2\pi} (4\cos y - (-4\cos y))\, dy$$

$$= 8\sin y \Big|_{3\pi/2}^{2\pi} = 0 - 8\sin\frac{3\pi}{2} = \underline{8}$$

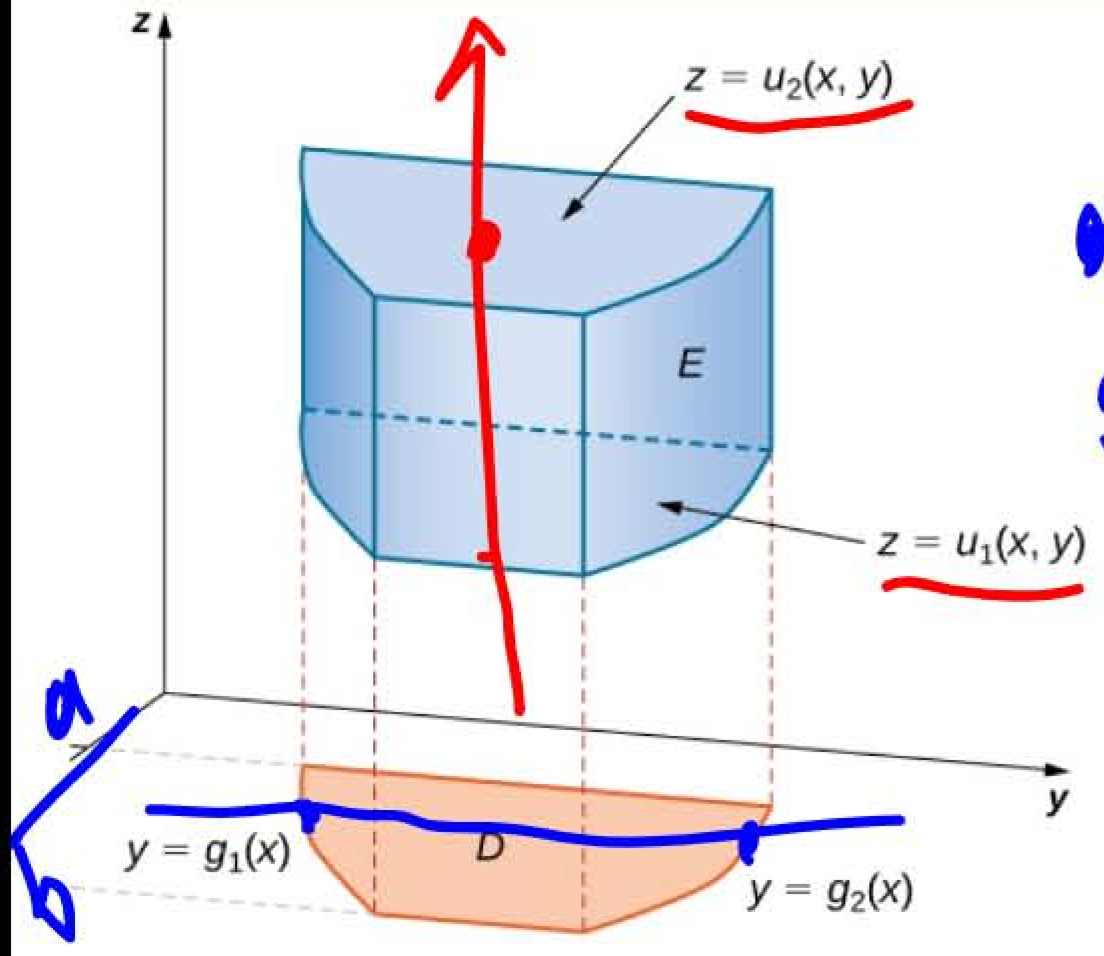
Theorem 5.10: Triple Integral over a General Region

The triple integral of a continuous function $f(x, y, z)$ over a general three-dimensional region

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

in $\mathbb{R}^3$, where D is the projection of E onto the xy-plane, is

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)dz \right] dA.$$



A box E where the projection D in the xy-plane is of

Then the triple integral becomes

$a \le x \le b$

$g_1(x) \le y \le g_2(x)$

$u_1(x,y) \le z \le u_2(x,y)$

$$\iiint_E f(x, y, z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)dz\,dy\,dx.$$

194. $\iiint\limits_E (xy + yz + xz)\,dV$, where

$E = \{(x, y, z) | 0 \le x \le 1,\ -x^2 \le y \le x^2,\ 0 \le z \le 1\}$

$$\int_0^1\int_0^1\int_{-x^2}^{x^2} (xy+yz+xz)\,dy\,dx\,dz = \int_0^1\int_0^1 \left(\frac{y^2}{2}(x+z)+xzy\right)\Bigg|_{-x^2}^{x^2} dx\,dz$$

$y(x+z)+xz$

$$\int_0^1\int_0^1 \left[\left(\cancel{\frac{x^4}{2}(x+z)}+x^3z\right) - \left(\cancel{\frac{x^4}{2}(x+z)} - x^3z\right)\right] dx\,dz$$

$$= \int_0^1\int_0^1 2x^3z\,dx\,dz = \int_0^1 \frac{x^4}{2}z\Bigg|_0^1 dz = \int_0^1 \frac{z}{2}\,dz = \frac{z^2}{4}\Bigg|_0^1 = \frac{1}{4}$$

an

5.25 Write five different iterated integrals equal to the given integral

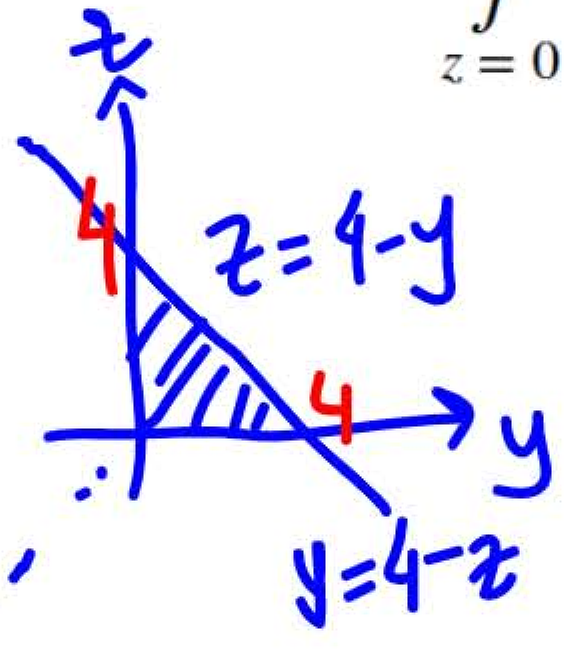
$$\int_{z=0}^{z=4}\int_{y=0}^{y=4-z}\int_{x=0}^{x=\sqrt{y}} f(x,\, y,\, z)dx\,dy\,dz.$$

D

$0 \leqslant z \leqslant 4$

$0 \leqslant y \leqslant 4-z$

$0 \leqslant x \leqslant \sqrt{y}$



$0 \leqslant y \leqslant 4$

$0 \leqslant z \leqslant 4-y$

$$\int_0^4 \int_0^{4-y} \int_0^{\sqrt{y}} \cdots dx\,dz\,dy$$

Theorem 5.11: Average Value of a Function of Three Variables

If $f(x, y, z)$ is integrable over a solid bounded region E with positive volume $V(E)$, then the average value of the function is

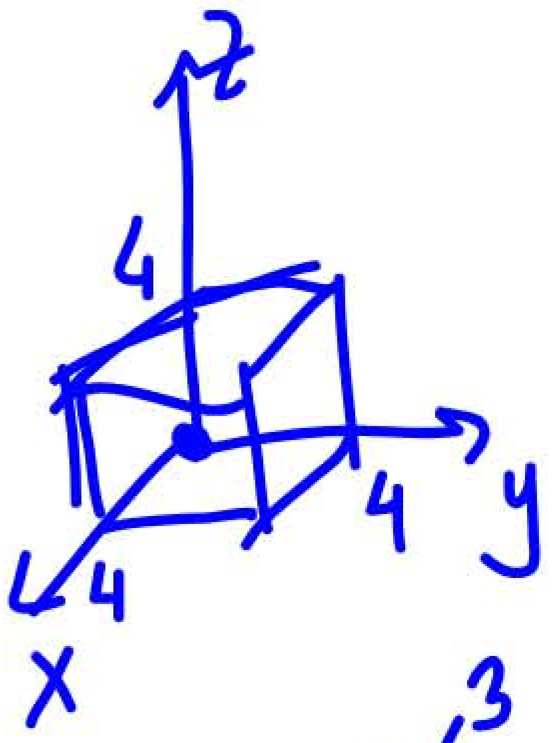
$$f_{\text{ave}} = \frac{1}{V(E)} \iiint_E f(x, y, z)\,dV.$$

Note that the volume is $V(E) = \iiint_E 1\,dV.$

$$f_{ave} = \frac{1}{A(R)} \iint_R f \ldots dA$$

$$A(R) = \iint_R 1 \, dA$$

5.26 Find the average value of the function $f(x, y, z) = xyz$ over the cube with sides of length 4 units in the first octant with one vertex at the origin and edges parallel to the coordinate axes.



$\text{Volume} = 4^3$

$$f_{ave} = \frac{1}{64}\int_0^4\int_0^4\int_0^4 xyz\,dx\,dy\,dz$$

$$\int_0^4\int_0^4 \frac{x^2}{2}yz\Big|_0^4\,dy\,dz = \int_0^4\int_0^4 8yz\,dy\,dz$$

$$= \int_0^4 4y^2z\Big|_0^4\,dz = \int_0^4 64z\,dz = 32z^2\Big|_0^4 = 32\times16$$

$$f_{ave} = \frac{1}{64}\,32\times16 = 8$$

In the following exercises, change the order of integration by integrating first with respect to z, then x, then y.

185. $\int_0^1 \int_1^2 \int_2^3 (x^2 + \ln y + z)\,dx\,dy\,dz = \int_1^2 \int_2^3 \int_0^1 (x^2 + \ln y + z)\,dz\,dx\,dy$

$$\int_1^2 \int_2^3 \left((x^2 + \ln y) z + \frac{z^2}{2} \right) \Big|_0^1 dx\,dy = \int_1^2 \int_2^3 \left(x^2 + \ln y + \frac{1}{2}\right) dx\,dy$$

$$= \int_1^2 \left(\frac{x^3}{3} + \left(\ln y + \frac{1}{2}\right) x \right) \Big|_2^3 dy = \int_1^2 \left(\frac{27-8}{3} + 3\ln y + \frac{3}{2} - 2\ln y - 1 \right) dy$$

$$\int_1^2 \left(\ln y + \frac{19}{3} + \frac{1}{2}\right) dy$$

(2) (3)

$$\int_1^2 \frac{41}{6}\, dy = \frac{41}{6} y \Big|_1^2 = \frac{41}{6}$$

$$\int_1^2 \ln y\, dy = y \ln y \Big|_1^2 - \int_1^2 y \cdot \frac{1}{y}\, dy = 2\ln 2 - 1 \cdot \ln 1 - (2-1)$$

$$= \underline{2\ln 2 - 1}$$

integration by parts

$u = \ln y \quad du = \frac{1}{y} dy$

$dv = dy \quad v = y$

$$\int \ln x\, dx = x \ln x - x + c$$

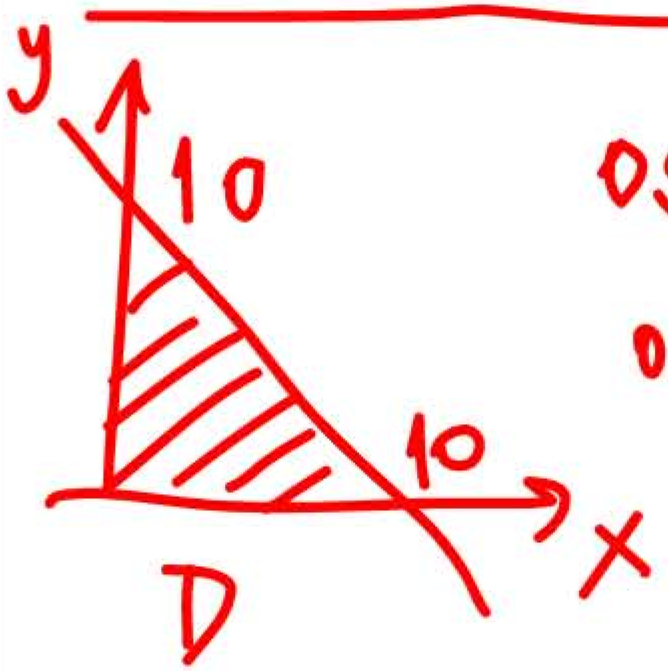
result.

$$\boxed{2\ln 2 - 1 + \frac{41}{6}}$$

$$2\ln 2 + 35/6$$

209. $$\iint_D \left(\int_0^{10-x-y} (x+2z)\,dz \right) dA,$$ where $D = \{(x, y) | y \ge 0,\ x \ge 0,\ x + y \le 10\}$

$$\iint_D (xz + z^2)\Big|_0^{10-x-y} dA$$



$0 \le x \le 10$

$0 \le y \le 10 - x$

$dA = dy\,dx$

$$\int_0^{10}\int_0^{10-x} \left(x(10-x-y) + (10-x-y)^2 \right) dy\,dx$$

OR. $0 \le y \le 10$

$0 \le x \le 10 - y$

$dA = dx\,dy$

$$\left(10xy - x^2y - \frac{y^2}{2}x + \frac{(10-x-y)^3}{-3} \right)\Bigg|_0^{10-x}$$

$$= \int_0^{10} \left(x(10-x)^2 - \frac{(10-x)^2}{2}x + \cancel{\frac{0}{-3}} - \left(\cancel{0} - \cancel{0} - \cancel{0} - \frac{(10-x)^3}{3} \right) \right) dx$$

$$= \int_0^{10} \left(50x - 10x^2 + x^3/2 + \frac{(10-x)^3}{3} \right) dx$$