4.5 | The Chain Rule

Calculus 1

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x).$$

$$(f(u))' = f'(u)\,u'$$

$y = f(x) \qquad x = g(t)$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$y \xrightarrow{\frac{dy}{dx}} x \xrightarrow{\frac{dx}{dt}} t$$

tree diagram

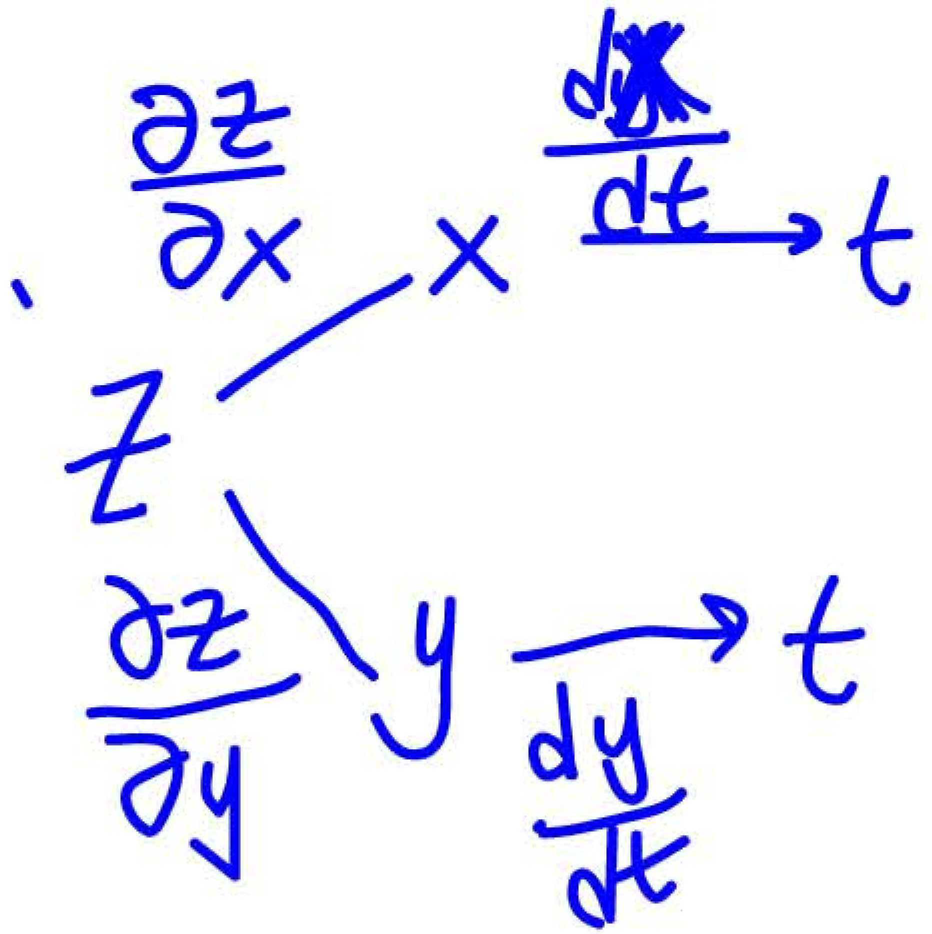
$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Theorem 4.8: Chain Rule for One Independent Variable (t)

Suppose that $x = g(t)$ and $y = h(t)$ are differentiable functions of t and $z = f(x, y)$ is a differentiable function of x and y. Then $z = f(x(t),\ y(t))$ is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}, \tag{4.29}$$

where the ordinary derivatives are evaluated at t and the partial derivatives are evaluated at $(x,\ y)$.



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228. Let $z = x^2 y$, where $x = t^2$ and $y = t^3$. Find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2xy \cdot 2t + x^2 \cdot 3t^2$$

$$= 2t^2 t^3 2t^1 + (t^2)^2 \cdot 3t^2$$

$$= 4t^6 + 3t^6$$

$$= 7t^6$$

direct substitution

$$z = (t^2)^2 t^3$$

$$= t^7$$

$$\frac{dz}{dt} = 7t^6$$

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4.23 Calculate dz/dt given the following functions. Express the final answer in terms of t.

$$z = f(x, y) = x^2 - 3xy + 2y^2,\ x = x(t) = 3\sin 2t,\ y = y(t) = 4\cos 2t$$

$$\frac{dz}{dt} = (2x - 3y)\,3\cos 2t \cdot 2 + (-3x + 4y)(-4\sin 2t)\cdot 2$$

$$= (6\sin 2t - 12\cos 2t)\,6\cos 2t + (16\cos 2t - 9\sin 2t)\cdot(-8\sin 2t).$$

$$= (36 - 128)\sin 2t\cos 2t - 72\cos^2 2t + 72\sin^2 2t$$

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tree diagram

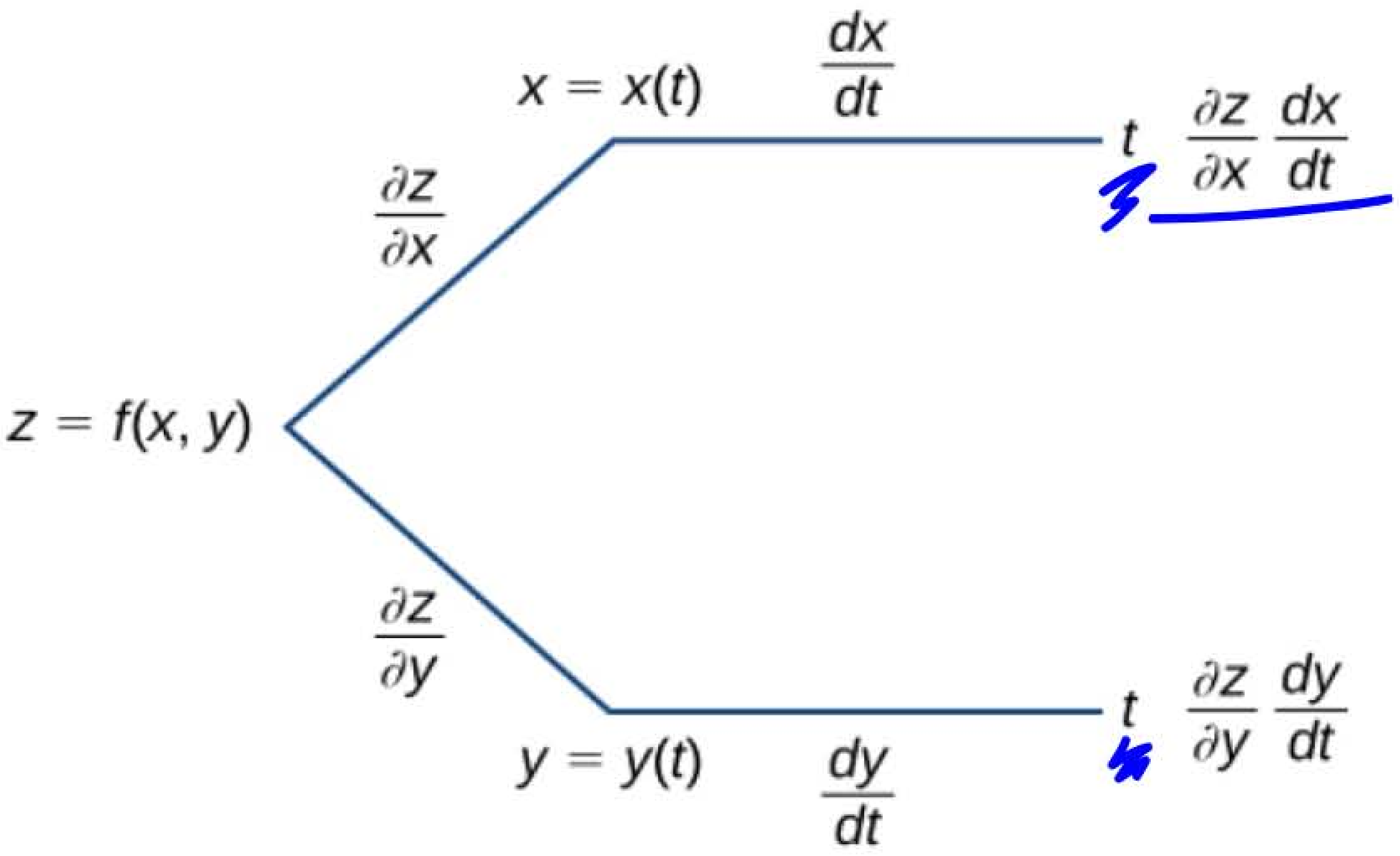


Figure 4.34 Tree diagram for the case

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

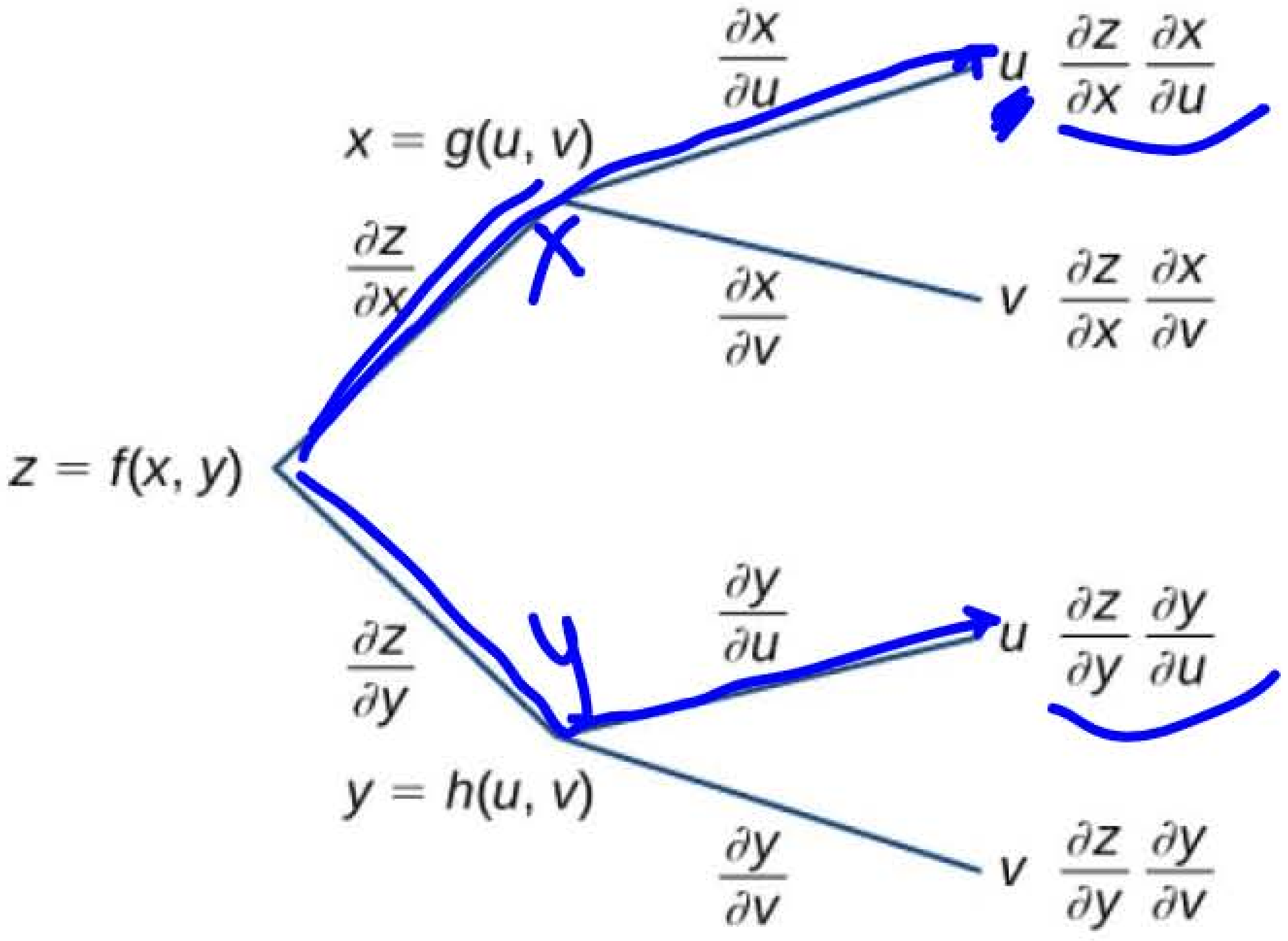
Theorem 4.9: Chain Rule for Two Independent Variables (u,v)

Suppose $x = g(u, v)$ and $y = h(u, v)$ are differentiable functions of u and v, and $z = f(x, y)$ is a differentiable function of x and y. Then, $z = f(g(u, v), h(u, v))$ is a differentiable function of u and v, and

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial x}{\partial u} \qquad (4.31)$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \qquad (4.32)$$



$z \quad x \quad y \qquad\qquad u \quad v$

216. Let $w(t, v) = e^{tv}$ where $t = r + s$ and $v = rs$.

Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial r} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial r}$$

$$= ve^{tv} \cdot 1 + te^{tv} \cdot s$$

$$\frac{\partial w}{\partial s} = ve^{tv} \cdot 1 + te^{tv} \cdot r$$

$$\frac{\partial t}{\partial s} = 1 \qquad \frac{\partial v}{\partial s} = r$$

4.24 Calculate $\partial z/\partial u$ and $\partial z/\partial v$ given the following functions:

$$z = f(x, y) = \frac{2x - y}{x + 3y}, \; x(u, v) = e^{2u} \cos 3v, \; y(u, v) = e^{2u} \sin 3v.$$

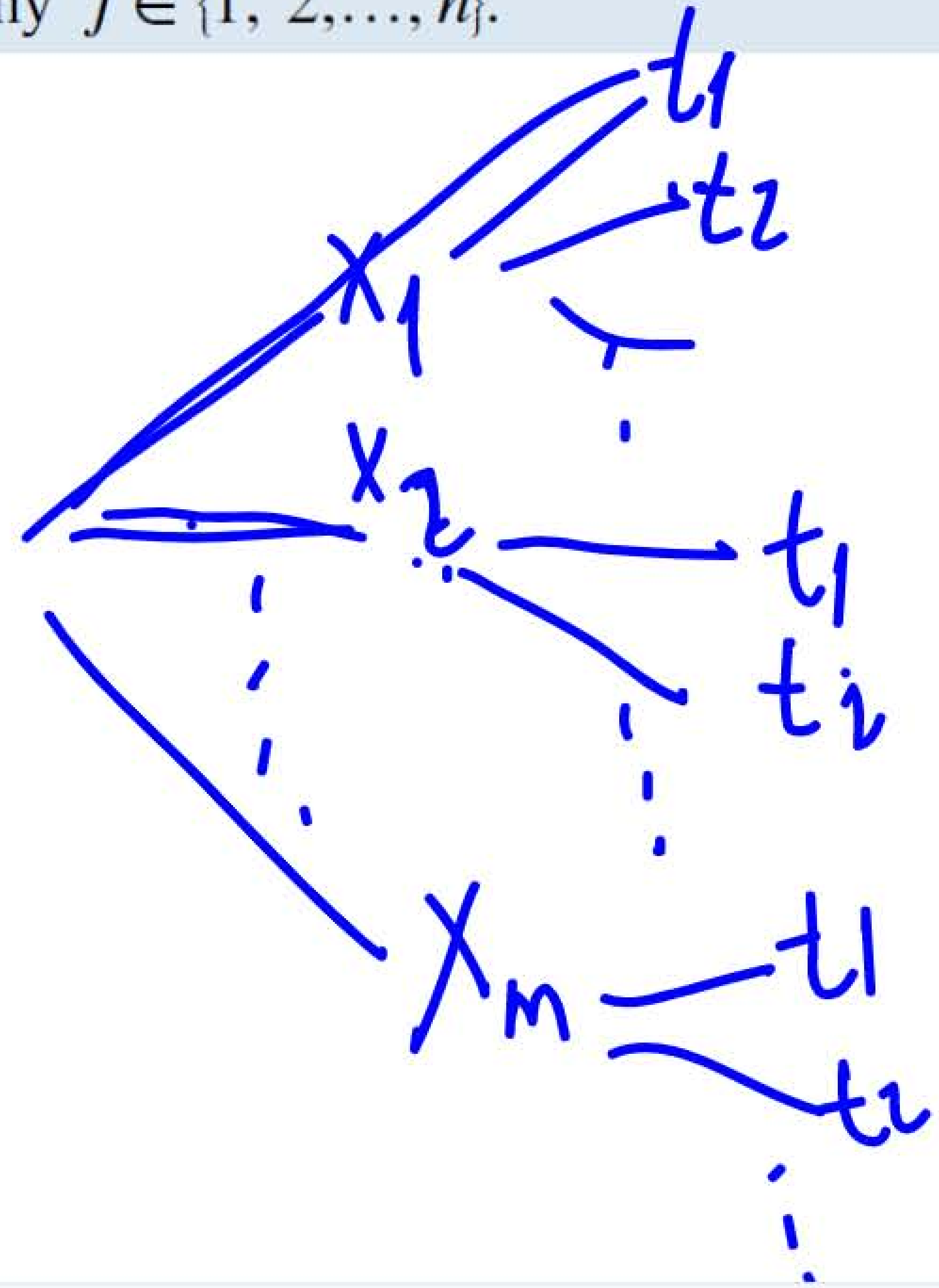
$\frac{\partial z}{\partial x}$ quotient rule

Theorem 4.10: Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m independent variables, and for each $i \in \{1, \ldots, m\}$, let $x_i = x_i(t_1, t_2, \ldots, t_n)$ be a differentiable function of n independent variables. Then

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m}\frac{\partial x_m}{\partial t_j} \qquad \textbf{(4.33)}$$

for any $j \in \{1, 2, \ldots, n\}$.



Implicit Differentiation

$$F = x^2 + xy + y^2 = 0$$

$$2x + 1 \cdot y + x \cdot y' + 2y\,y' = 0$$

$$y'(x+2y) = -2x - y$$

$$\frac{dy}{dx} = \frac{-2x-y}{x+2y}$$

alternative

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

partial

$$F_x = \frac{\partial F}{\delta x}$$

Theorem 4.11: Implicit Differentiation of a Function of Two or More Variables

Suppose the function $z = f(x, y)$ defines y implicitly as a function $y = g(x)$ of x via the equation $f(x, y) = 0$. Then

$$y' = \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \qquad \text{(4.34)}$$

provided $f_y(x, y) \neq 0$.

If the equation $f(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then

$$\frac{\partial z}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{\partial f/\partial y}{\partial f/\partial z} \qquad \text{(4.35)}$$

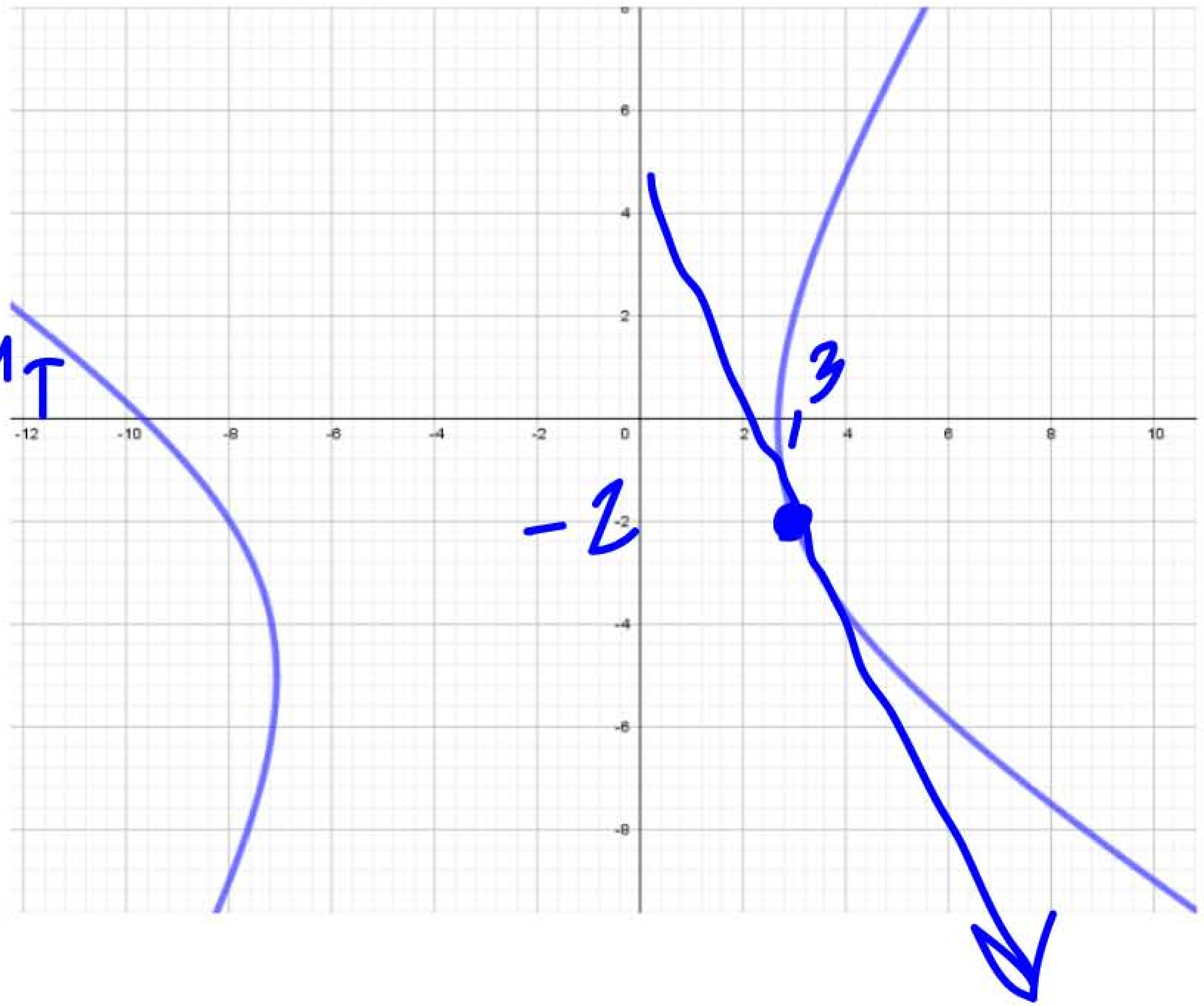
as long as $f_z(x, y, z) \neq 0$.

4.27 Find dy/dx if y is defined implicitly as a function of x by the equation $x^2 + xy - y^2 + 7x - 3y - 26 = 0$. What is the equation of the tangent line to the graph of this curve at point $(3, -2)$?

f

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x+y+7}{x-2y-3}$$

$$\left.\frac{dy}{dx}\right|_{(3,-2)} = -\frac{6-2+7}{3+4-3} = -\frac{11}{4} = m_T$$

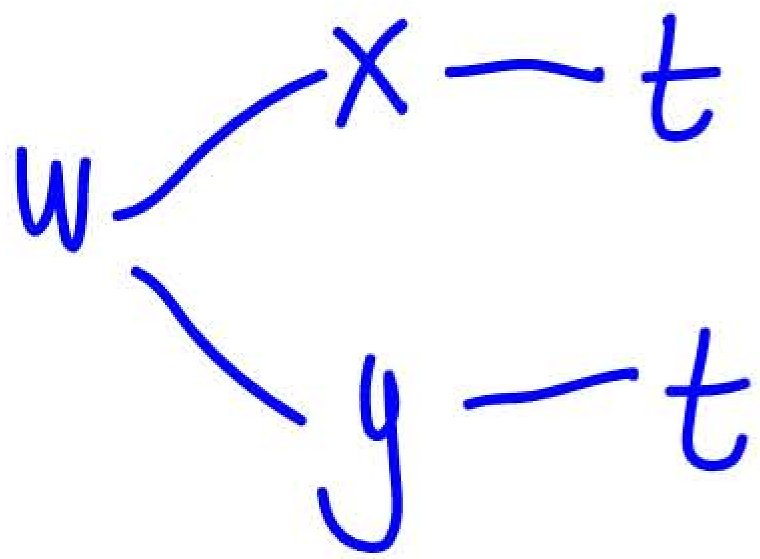


$$y-(-2) = -\frac{11}{4}(x-3)$$

$$y = -\frac{11x}{4} + \frac{25}{4}$$

218. If $w = xy^2$, $x = 5\cos(2t)$, and $y = 5\sin(2t)$, find $\frac{\partial w}{\partial t}$.

$$= y^2 \cdot (-5\sin 2t \cdot 2) + 2xy(5\cos 2t) \cdot 2$$



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

For the following exercises, find $\frac{dy}{dx}$ using partial derivatives.

238. $x^2 y^3 + \cos y = 0$

implicit

$f(x,y) = 0$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2xy^3}{3x^2y^2 - \sin y}$$

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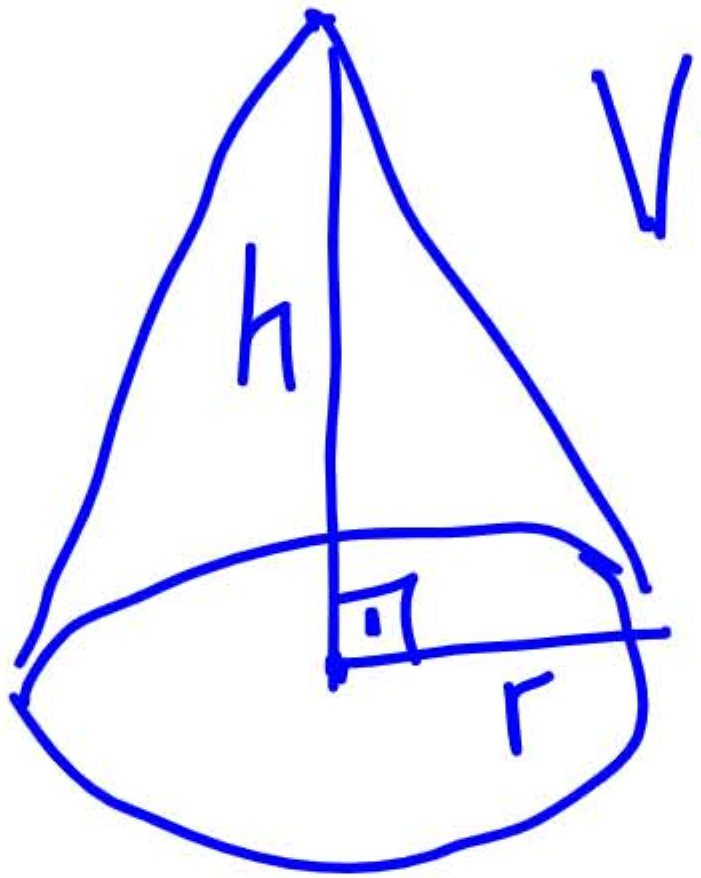
253. The radius of a right circular cone is increasing at 3 cm/min whereas the height of the cone is decreasing at 2 cm/min. Find the rate of change of the volume of the cone when the radius is 13 cm and the height is 18 cm.

$$\frac{dr}{dt} = 3\ \text{cm/min}$$

$$\frac{dh}{dt} = -2\ \text{cm/min}$$



$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

$$= \frac{2\pi}{3} r h \cdot 3 + \frac{\pi r^2}{3}\cdot(-2)$$

$$= 2\pi \times 13 \times 18 - \frac{2}{3}\pi(13)^2 = \underline{\qquad} \ \text{cm}^3/\text{min}$$

For the following exercises, find $\frac{df}{dt}$ using the chain rule and direct substitution.

222. $f(x, y) = \sqrt{x^2 + y^2}$, $y = t^2$, $x = t$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{x}{\sqrt{x^2+y^2}} \cdot 1 + \frac{y}{\sqrt{x^2+y^2}} \cdot 2t = \frac{t+2t^3}{\sqrt{t^2+t^4}}$$

$$\frac{\partial f}{\partial x} = \frac{\partial (x^2+y^2)^{1/2}}{\partial x} = \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \cdot 2x$$

$$f(x,y) = \sqrt{t^2+t^4}$$

$$\frac{df}{dt} = \frac{1}{2}(t^2+t^4)^{-\frac{1}{2}} \cdot (2t+4t^3)$$

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