4.6 | Directional Derivatives and the Gradient

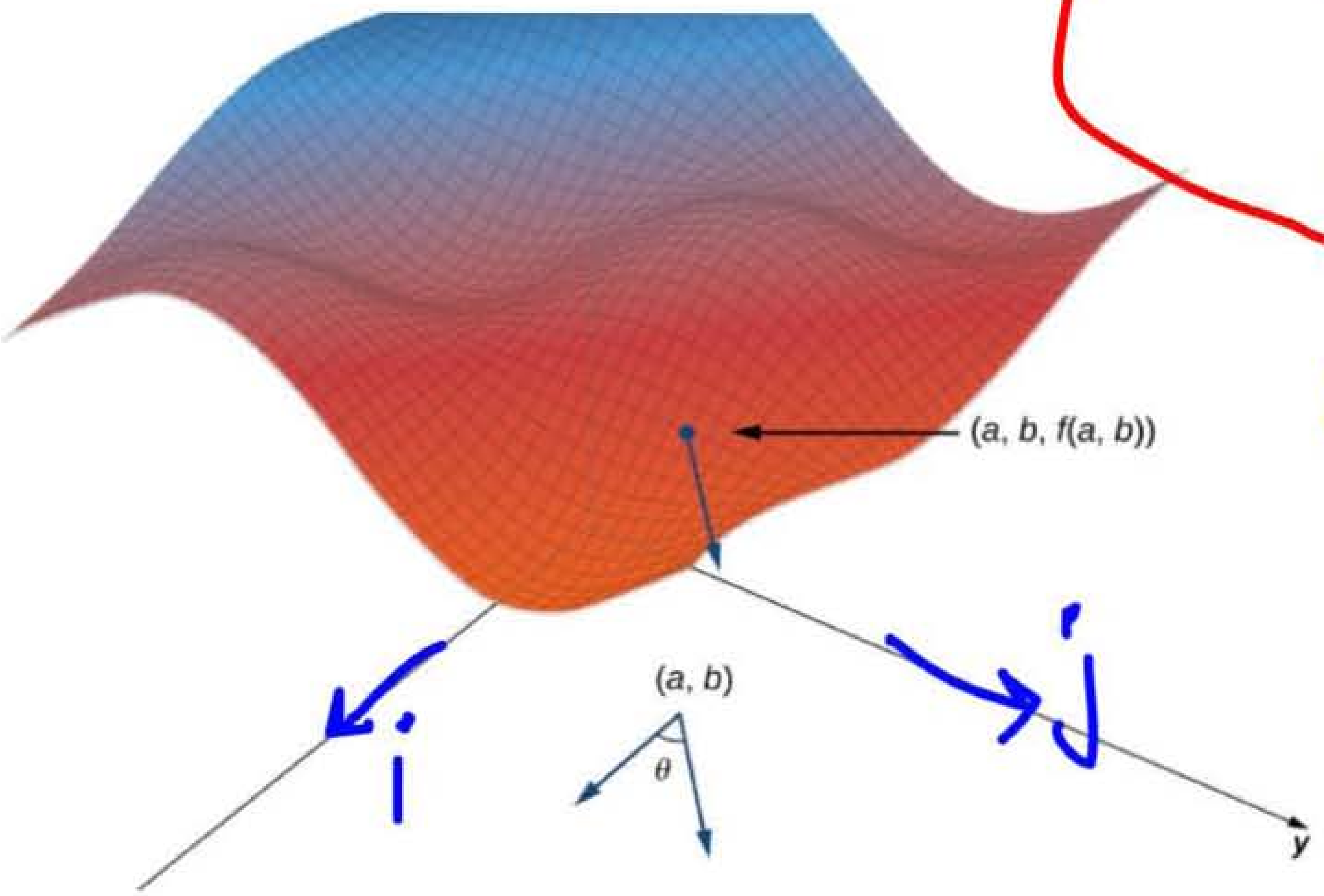
Chain rule $z = f(x,y)$ $x = x(t)$ $y(t)$

$$z \begin{cases} x - t \\ y \to t \end{cases}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$D_u f = \underbrace{\langle f_x, f_y \rangle}_{\text{gradient}} \cdot \vec{u}$$

unit vector



$$\nabla f = \langle f_x, f_y \rangle$$

Three-Dimensional Gradients and Directional Derivatives

The definition of a gradient can be extended to functions of more than two variables.

Definition

Let $w = f(x, y, z)$ be a function of three variables such that f_x, f_y, and f_z exist. The vector $\nabla f(x, y, z)$ is called the gradient of f and is defined as

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}. \tag{4.40}$$

$\nabla f(x, y, z)$ can also be written as grad $f(x, y, z)$.

$\nabla f = \langle f_x, f_y, f_z \rangle$

281. Find the gradient of $f(x, y, z) = xy + yz + xz$ at point $P(1, 2, 3)$.

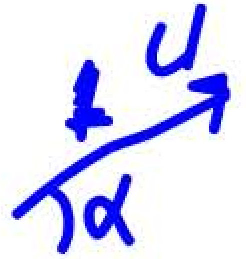
$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle y+z, x+z, y+x \rangle = \langle 2+3, 1+3, 2+1 \rangle$$
$$= \langle 5, 4, 3 \rangle$$

Theorem 4.15: Directional Derivative of a Function of Three Variables

Let $f(x, y, z)$ be a differentiable function of three variables and let $\mathbf{u} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$ be a unit vector. Then, the directional derivative of f in the direction of $\mathbf{u}$ is given by

unit vector

$$\begin{aligned} D_{\mathbf{u}} f(x, y, z) &= \nabla f(x, y, z) \cdot \mathbf{u} \\ &= f_x(x, y, z)\cos\alpha + f_y(x, y, z)\cos\beta + f_z(x, y, z)\cos\gamma. \end{aligned} \tag{4.42}$$



4.33 Calculate $D_{\mathbf{u}} f(x, y, z)$ and $D_{\mathbf{u}} f(0, -2, 5)$ in the direction of $\mathbf{v} = -3\mathbf{i} + 12\mathbf{j} - 4\mathbf{k}$ for the function $f(x, y, z) = 3x^2 + xy - 2y^2 + 4yz - z^2 + 2xz$.

$$\|\vec{v}\| = \sqrt{(-3)^2 + 12^2 + (-4)^2} = 13$$

$$u = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle -3, 12, -4 \rangle}{13}$$

unit

$$u = \left\langle -\frac{3}{13}, \frac{12}{13}, -\frac{4}{13} \right\rangle$$

$x - 4y + 4z$

$$D_u f = \langle 6x + y + 2z,\ x - 4\cancel{y} + \cancel{4y}\,z,\ 4y - 2z + 2x \rangle \cdot \vec{u}$$

Substitute $(0, -2, 5)$

$$\langle 8, 28, -18 \rangle \cdot \vec{u}$$

4.7 | Maxima/Minima Problems

Learning Objectives

4.7.1 Use partial derivatives to locate critical points for a function of two variables.

4.7.2 Apply a second derivative test to identify a critical point as a local maximum, local minimum, or saddle point for a function of two variables.

4.7.3 Examine critical points and boundary points to find absolute maximum and minimum values for a function of two variables.

Definition

Let $z = f(x, y)$ be a function of two variables that is defined on an open set containing the point (x_0, y_0). The point (x_0, y_0) is called a **critical point of a function of two variables** f if one of the two following conditions holds:

1. $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$

OR

2. Either $f_x(x_0, y_0)$ or $f_y(x_0, y_0)$ does not exist.

$f' = 0$

or f' does not exist

4.34 Find the critical point of the function $f(x, y) = x^3 + 2xy - 2x - 4y$.

$f_x = 3x_0^2 + 2y_0 - 2 = 0$

$12 + 2y_0 - 2 = 0$

$2y_0 = -10$

$y_0 = -5$

$f_y = 2x_0 - 4 = 0 \rightarrow x_0 = 2$

$(2, -5)$ is a critical point.

$z = \frac{\ldots}{x - y}$ $y = x$ $(1, 1)$

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Definition

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Then f has a *local maximum* at (x_0, y_0) if

$$f(x_0, y_0) \geq f(x, y)$$

for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a *local maximum value*. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a *global maximum* (also called an *absolute maximum*) at (x_0, y_0).

The function f has a *local minimum* at (x_0, y_0) if

$$f(x_0, y_0) \leq f(x, y)$$

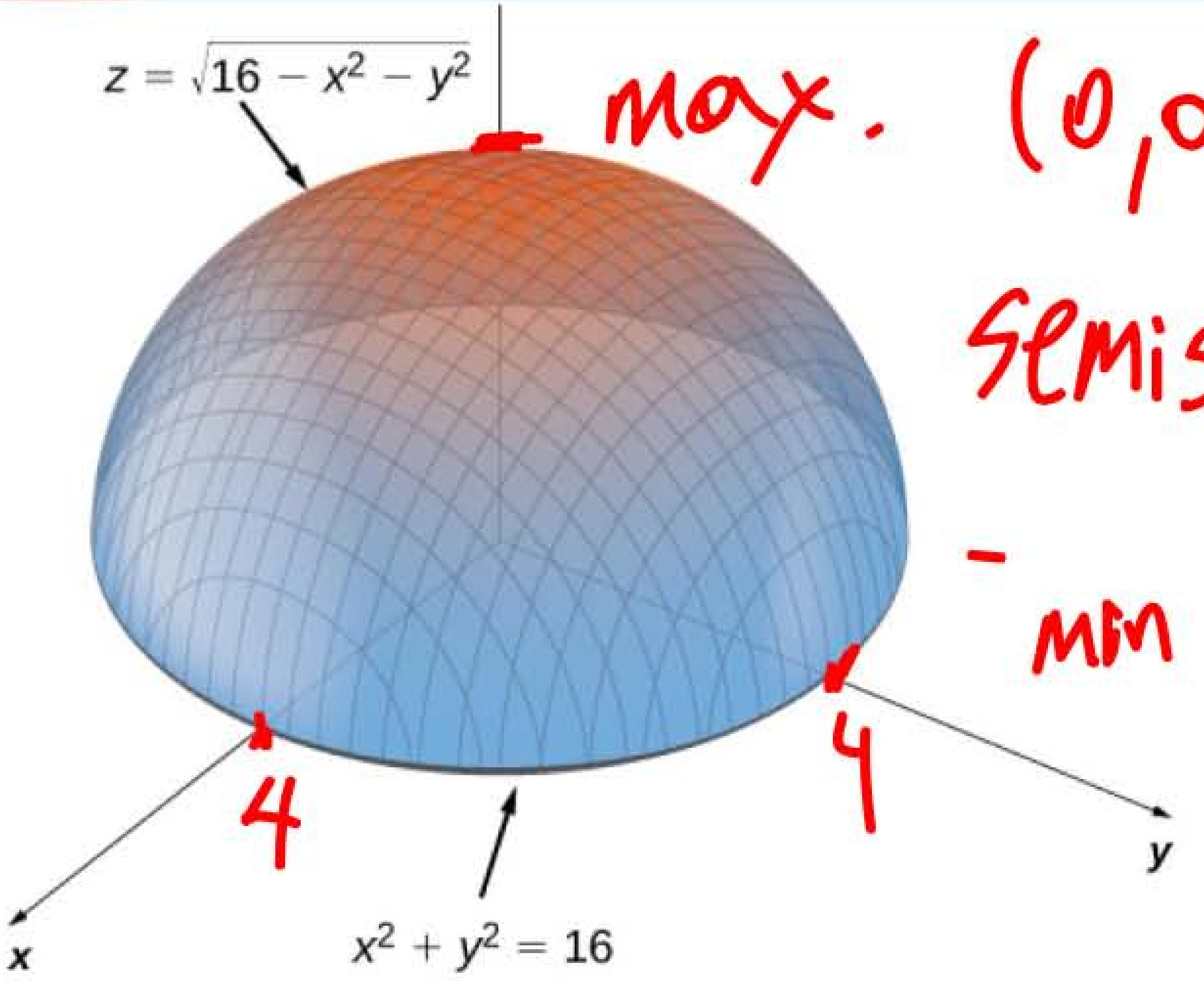
for all points (x, y) within some disk centered at (x_0, y_0). The number $f(x_0, y_0)$ is called a *local minimum value*. If the preceding inequality holds for every point (x, y) in the domain of f, then f has a *global minimum* (also called an *absolute minimum*) at (x_0, y_0).

If $f(x_0, y_0)$ is either a local maximum or local minimum value, then it is called a *local extremum* (see the following figure).

Theorem 4.16: Fermat's Theorem for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables that is defined and continuous on an open set containing the point (x_0, y_0). Suppose f_x and f_y each exists at (x_0, y_0). If f has a local extremum at (x_0, y_0), then (x_0, y_0) is a critical point of f.

(max, min)



max. (0, 0, 4) max → 4

semisphere

min → 0 = z

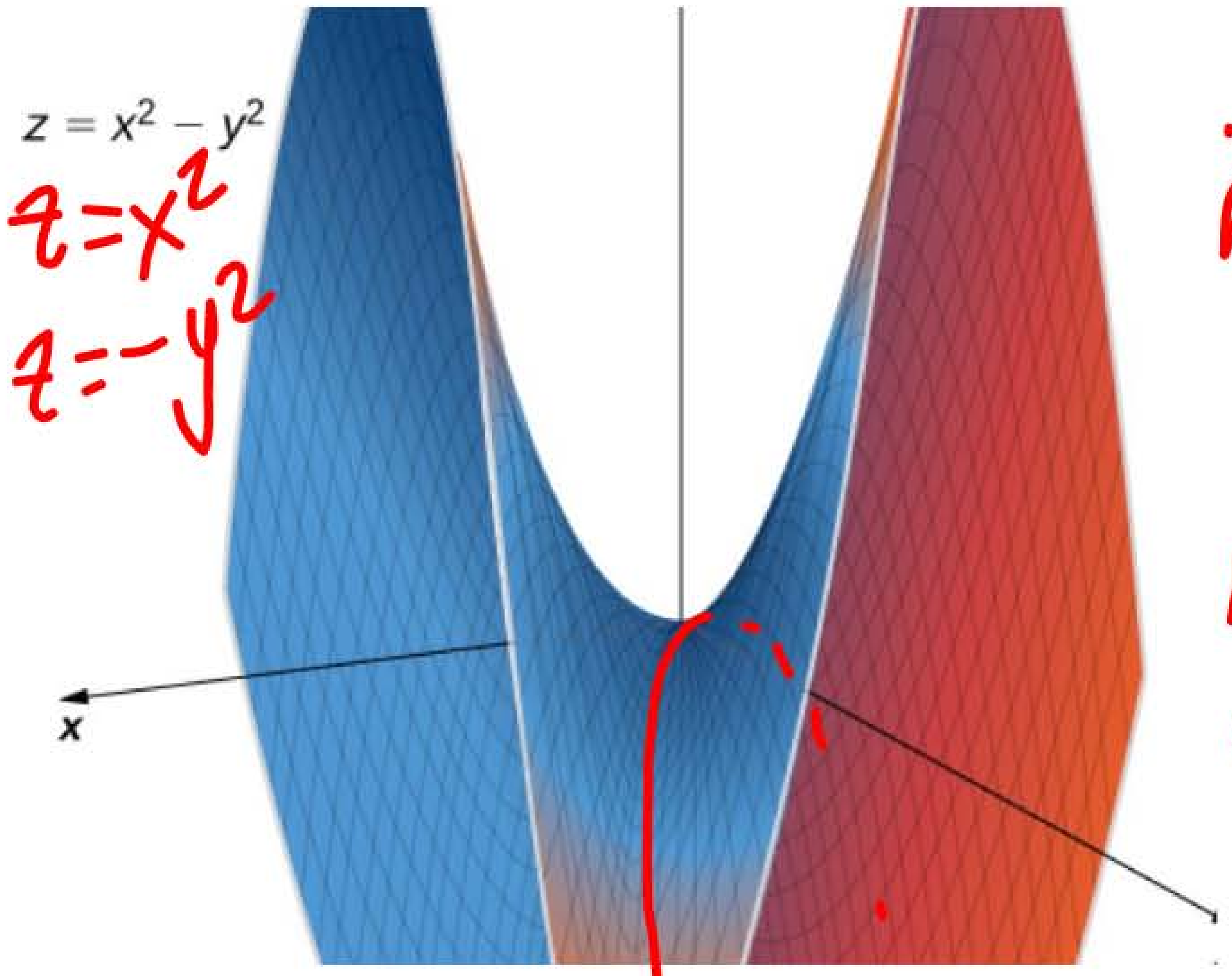
Figure 4.47 The graph of $z = \sqrt{16 - x^2 - y^2}$ has a maximum value when $(x, y) = (0, 0)$. It attains its minimum value at the boundary of its domain, which is the circle $x^2 + y^2 = 16$.

Definition

pelana

Given the function $z = f(x, y)$, the point $\big(x_0, y_0, f(x_0, y_0)\big)$ is a saddle point if both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, but f does not have a local extremum at (x_0, y_0).

(similar to inflection point)



$z = x^2$

$z = -y^2$

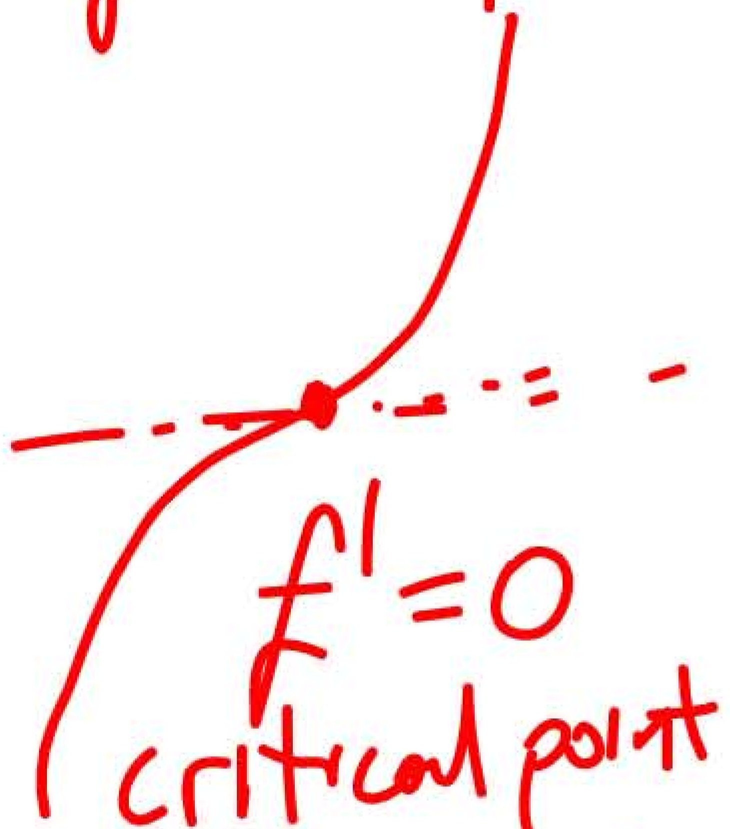
$f_x = 0 = f_y$

a critical point

not max or min

we call it

Saddle point



$f' = 0$

critical point

not max, min

inflection point.

Second Derivative Test

discriminant D

Theorem 4.17: Second Derivative Test

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). Suppose $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$. Define the quantity

critical point

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2. \qquad (4.43)$$

i. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum at (x_0, y_0).

ii. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum at (x_0, y_0).

iii. If $D < 0$, , then f has a saddle point at (x_0, y_0).

iv. If $D = 0$, then the test is inconclusive.

Problem-Solving Strategy: Using the Second Derivative Test for Functions of Two Variables

Let $z = f(x, y)$ be a function of two variables for which the first- and second-order partial derivatives are continuous on some disk containing the point (x_0, y_0). To apply the second derivative test to find local extrema, use the following steps:

1. Determine the critical points (x_0, y_0) of the function f where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Discard any points where at least one of the partial derivatives does not exist.
2. Calculate the discriminant $D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - \left(f_{xy}(x_0, y_0)\right)^2$ for each critical point of f.
3. Apply **Second Derivative Test** to determine whether each critical point is a local maximum, local minimum, or saddle point, or whether the theorem is inconclusive.

$D > 0$ $f_{xx} > 0$ minimum

$D > 0$ $f_{xx} < 0$ maximum.

$D < 0$ saddle

$D = 0$ inconclusive

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4.35 Use the second derivative to find the local extrema of the function

$$f(x, y) = x^3 + 2xy - 6x - 4y^2.$$

$f_x = 3x^2 + 2y - 6 = 0$

$f_y = 2x - 8y = 0 \Rightarrow x = 4y$ subst. $3(4y)^2 + 2y - 6 = 0$

$24y^2 + y - 3 = 0$

$y = 6 - 3x^2$

$(8y + 3)(3y - 1) = 0$

$2x - 8(6 - 3x^2) = 0$

$y = -\frac{3}{8}$ OR $y = \frac{1}{3}$

$12x^2 + x - 24 = 0$

$x = -\frac{3}{2}$ $x = \frac{4}{3}$ critical points

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Absolute Maxima and Minima

Theorem 4.19: Finding Extreme Values of a Function of Two Variables

Assume $z = f(x, y)$ is a differentiable function of two variables defined on a closed, bounded set D. Then f will attain the absolute maximum value and the absolute minimum value, which are, respectively, the largest and smallest values found among the following:

calculus 1 → critical $[3,5]$

i. The values of f at the critical points of f in D. region

ii. The values of f on the boundary of D. $\boxed{D}$ D

Problem-Solving Strategy: Finding Absolute Maximum and Minimum Values

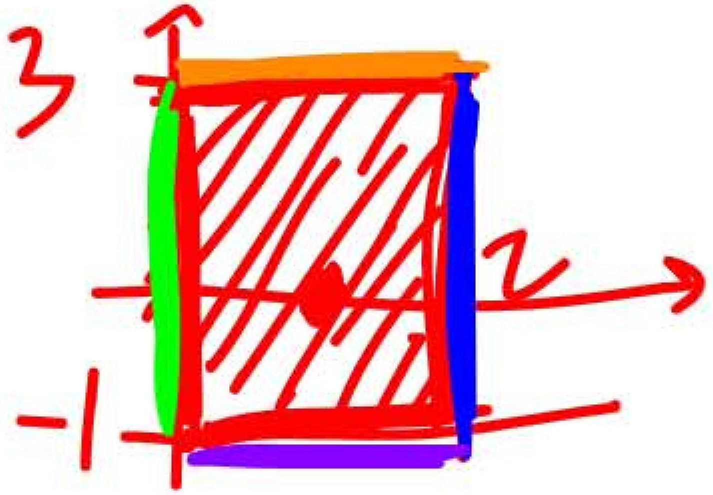
Let $z = f(x, y)$ be a continuous function of two variables defined on a closed, bounded set D, and assume f is differentiable on D. To find the absolute maximum and minimum values of f on D, do the following:

1. Determine the critical points of f in D.
2. Calculate f at each of these critical points.
3. Determine the maximum and minimum values of f on the boundary of its domain.
4. The maximum and minimum values of f will occur at one of the values obtained in steps 2 and 3.

4.36 Use the problem-solving strategy for finding absolute extrema of a function to find the absolute extrema of the function

$$f(x, y) = 4x^2 - 2xy + 6y^2 - 8x + 2y + 3$$

on the domain defined by $0 \le x \le 2$ and $-1 \le y \le 3$.



$f_x = 8x - 2y - 8 = 0$

$f_y = -2x + 12y + 2 = 0$

$48x - \cancel{12}y = 48$

$+ \; -2x + \cancel{12}y = -2$

$46x = 46$

$\boxed{x = 1, \; y = 0}$

$z = f(1,0) = 4 - 0 - 8 + 3 = -1$ MIN

$x = 0 \quad y = t, \quad -1 \le t \le 3 \quad z = 6t^2 + 2t + 3$

$-\frac{b}{2a} = -\frac{2}{2 \times 6} = -\frac{1}{6} \quad y = -\frac{1}{6} \quad z = \frac{1}{6} - \frac{2}{6} + 3 = 19/6$

$z = 6t^2 + 2t + 3, \quad -1 \leq t \leq 3$

$z = 6 - 2 + 3 = 7$

$z = 54 + 6 + 3 = 63$ MAX

$y = -1, \; x = t, \; 0 \leq t \leq 2$

$z = 4t^2 + 2\cancel{t} + 6 - 8t + 1$

$z = 4t^2 - 6t + 7$

$-\frac{b}{2a} = -\frac{-6}{2 \times 4} = \frac{3}{4}$

$z\left(\frac{3}{4}\right) = \frac{9}{4} - \frac{18}{4} + 7 = \frac{19}{4}$

$x = 2, \; y = t, \; -1 \leq t \leq 3$

$z = 1\cancel{6} - 4t + 6t^2 - 1\cancel{6} + 2t + 3$

$z = 6t^2 - 2t + 3$

$-\frac{b}{2a} = -\frac{-2}{2 \times 6} = \frac{1}{6} \qquad z = \frac{1}{6} - \frac{2}{6} + 3 = \frac{17}{6}$

$z(-1) = 6 + 2 + 3 = 11 \qquad z(3) = 54 - 6 + 3 = 51$

$y = 3, \; x = t, \; 0 \leq t \leq 2$

$z = 4t^2 - 6t + 54 - 8t + 9$

$= 4t^2 - 14t + 63 \qquad -\frac{-14}{8} = \frac{7}{4}$

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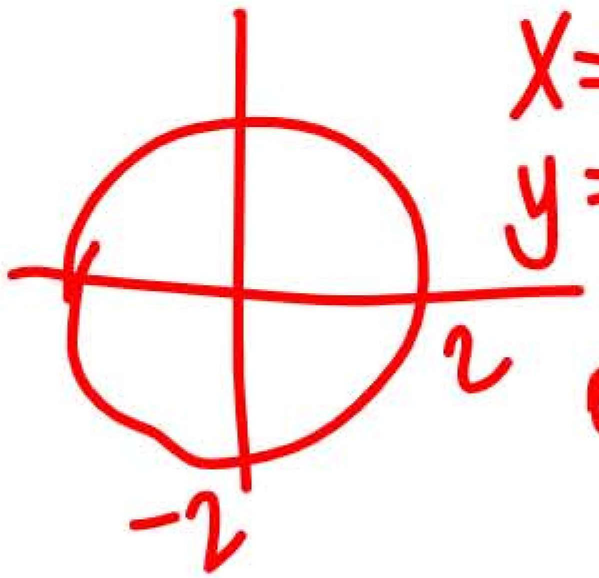
345. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2y + 1$ on the region $R = \{(x, y) | x^2 + y^2 \leq 4\}$.

$f_x = 2x = 0$

$f_y = 2y - 2 = 0$

$(0, 1)$ is the critical point

$z = f(0,1) = 0 + 1^2 - 2 + 1 = 0$ MIN



$x = 2\cos t$

$y = 2\sin t$

$0 \leq t \leq 2\pi$

$z = 4\cos^2 t + 4\sin^2 t - 4\sin t + 1$

$z = 5 - 4\sin t, \quad 0 \leq t < 2\pi$

$-4\cos t = 0 \quad t = \frac{\pi}{2}, \frac{3\pi}{2}$ critical

$z(0) = 5$

$z(2\pi) = 5$

$z\left(\frac{\pi}{2}\right) = 5 - 4 = 1$

$z\left(\frac{3\pi}{2}\right) = 5 + 4 = 9$ MAX

4.8 | Lagrange Multipliers

Theorem 4.20: Method of Lagrange Multipliers: One Constraint

Let f and g be functions of two variables with continuous partial derivatives at every point of some open set containing the smooth curve $g(x, y) = 0$. Suppose that f, when restricted to points on the curve $g(x, y) = 0$, has a local extremum at the point (x_0, y_0) and that $\nabla g(x_0, y_0) \neq 0$. Then there is a number λ called a **Lagrange multiplier**, for which

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Problem-Solving Strategy: Steps for Using Lagrange Multipliers

1. Determine the objective function $f(x, y)$ and the constraint function $g(x, y)$. Does the optimization problem involve maximizing or minimizing the objective function?
2. Set up a system of equations using the following template:

$$\begin{aligned} \nabla f(x_0, y_0) &= \lambda \nabla g(x_0, y_0) \\ g(x_0, y_0) &= 0. \end{aligned}$$

3. Solve for x_0 and y_0.
4. The largest of the values of f at the solutions found in step 3 maximizes f; the smallest of those values minimizes f.

4.37 Use the method of Lagrange multipliers to find the maximum value of $f(x, y) = 9x^2 + 36xy - 4y^2 - 18x - 8y$ subject to the constraint $3x + 4y = 32$.

$\nabla f = \langle 18x + 36y - 18, 36x - 8y - 8 \rangle$

$g(x,y) = 3x + 4y - 32 = 0$

$\nabla g = \langle 3, 4 \rangle$

$\boxed{\nabla f = \lambda \nabla g, \quad g(x,y) = 0}$

$18x_0 + 36y_0 - 18 = \lambda 3$

$36x_0 - 8y_0 - 8 = \lambda 4$

$3x_0 + y_0 = 32$

$72x_0 + 144y_0 - 72 = 12\lambda$

$= 108x_0 - 24y_0 - 24$

$168y_0 - 36x_0 = 48$

$14y_0 - 3x_0 = 4$

$+ \quad 3x_0 + y_0 = 32$

$15y_0 = 36$

$y_0 = \frac{12}{5}$

$x_0 = \frac{148}{5 \times 3} = \frac{148}{15}$

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$f_{xx} = 6x \quad f_{xy} = 2$

$f_{yy} = -8$

discriminant $D = 6x \cdot (-8) - (2)^2$

$= -48x - 4$

$\left(\frac{4}{3}, \frac{1}{3}\right) \rightarrow D = -48 \cdot \frac{4}{3} - 4 = -68 < 0$

saddle point

$\left(-\frac{3}{2}, -\frac{3}{8}\right) \rightarrow D = -48 \cdot \left(-\frac{3}{2}\right) - 4 = 72 - 4 = 68 > 0$

local maximum

$f_{xx} = 6x = 6\left(-\frac{3}{2}\right) = -9 < 0$ max

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100%