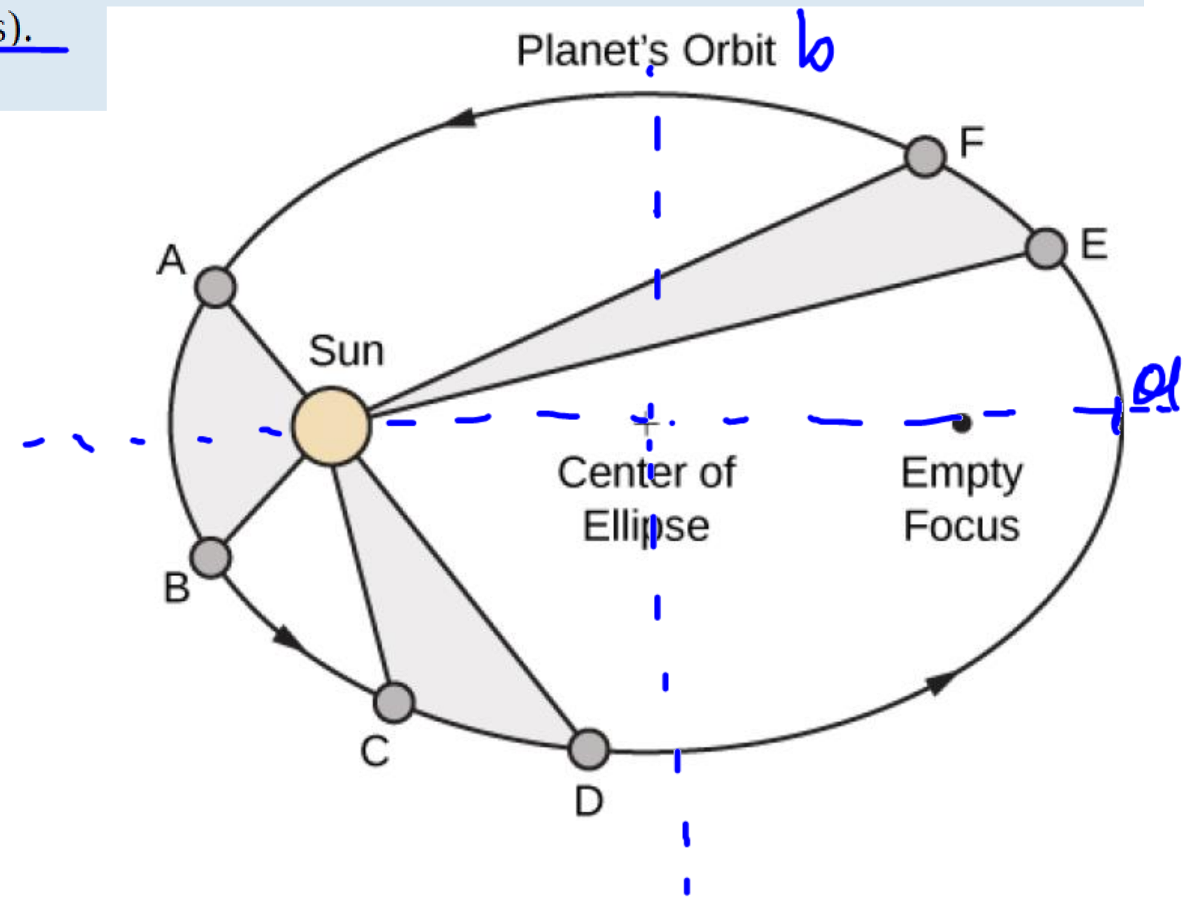
Theorem 3.9: Kepler's Laws of Planetary Motion

i. The path of any planet about the Sun is elliptical in shape, with the center of the Sun located at one focus of the ellipse (the law of ellipses).

ii. A line drawn from the center of the Sun to the center of a planet sweeps out equal areas in equal time intervals (the law of equal areas) (**Figure 3.18**).

iii. The ratio of the squares of the periods of any two planets is equal to the ratio of the cubes of the lengths of their semimajor orbital axes (the law of harmonies).

$$\frac{P_1^2}{P_2^2} = \frac{a_1^3}{a_2^3}$$



3.17 Titan is the largest moon of Saturn. The mass of Titan is approximately m 1.35×10^{23} kg. The mass of Saturn is approximately M 5.68×10^{26} kg. Titan takes approximately 16 days to orbit Saturn. Use this information, along with the universal gravitation constant $G = 6.67\times10^{-11}$ m /kg · sec^2 to estimate the distance from Titan to Saturn.

$$P^2 = \frac{4\pi^2 a^3}{G(m+M)}$$

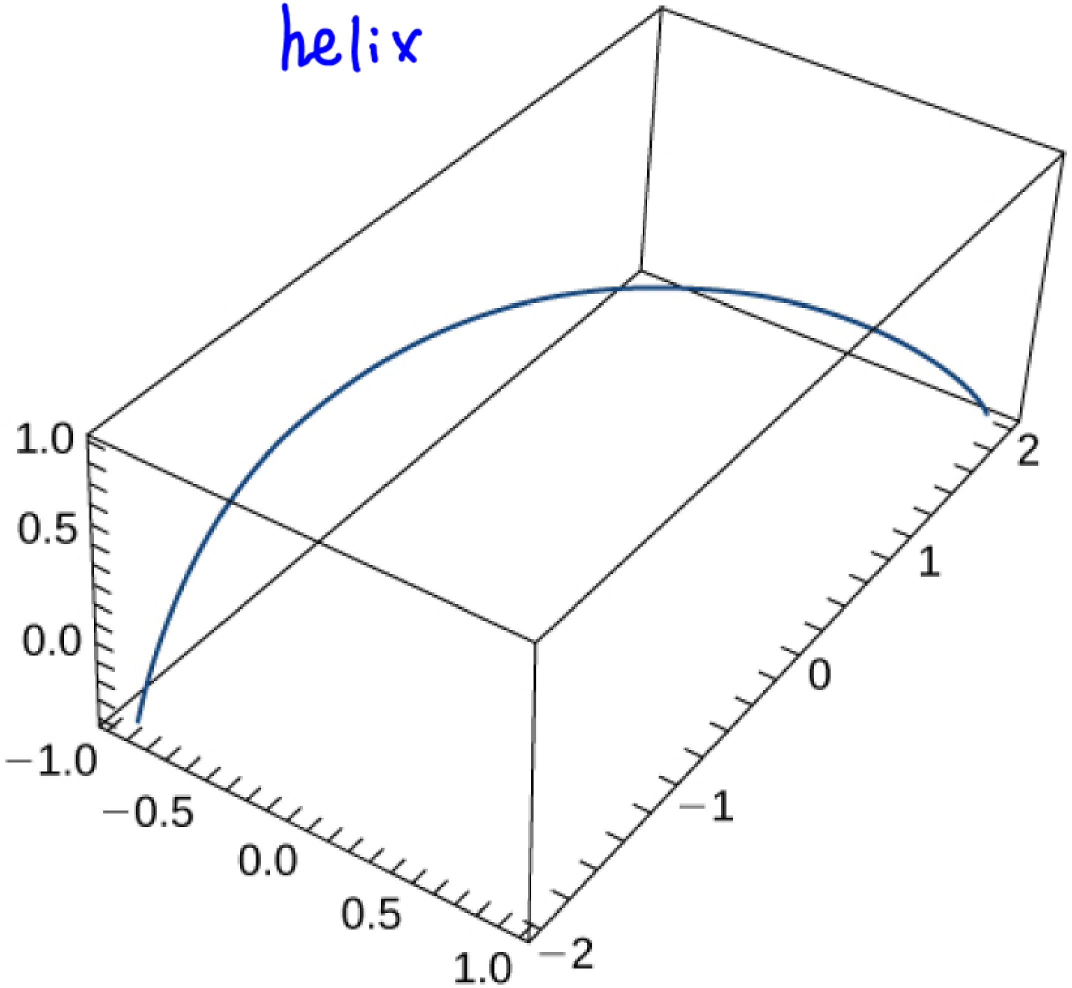
$P = 16$ days $= 16 \times 24 \times 3600$ sec

$a = ?$

Find the velocity, acceleration, and speed of a particle with the given position function.

162. $\mathbf{r}(t) = \langle \sin t, t, \cos t \rangle$. The graph is shown here:

helix



$v(t) = r'(t) = \langle \cos t, 1, -\sin t \rangle$

$a(t) = v'(t) = \langle -\sin t, 0, -\cos t \rangle$

$v(t) = \| v(t) \|$

$= \sqrt{\cos^2 t + 1^2 + \sin^2 t}$

$= \sqrt{2}$

163. The position function of an object is given by $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle$. At what time is the speed a minimum?

$$\vec{v}(t) = \langle 2t, 5, 2t - 16 \rangle$$

speed is a scalar function

$$v(t) = \|\vec{v}(t)\| = \sqrt{4t^2 + 25 + 4(t-8)^2}$$

minimum?

$$s(t) = \sqrt{8t^2 - 64t + 281}$$

$$(\sqrt{u})' = \frac{u'}{2\sqrt{u}}$$

$$s'(t) = \frac{16t - 64}{2\sqrt{8t^2 - 64t + 281}} = 0$$

$s'(4) = 0$ critical value

4

s' — ϕ +

s

min

Minimum speed

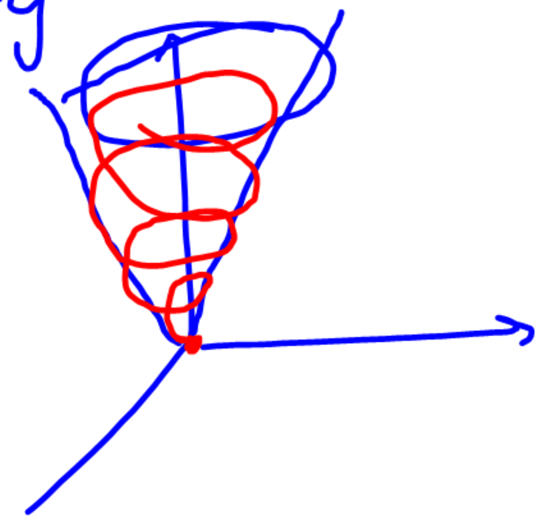
$$s(4) = \sqrt{128 - 256 + 281}$$

$$= \sqrt{153}$$

m

Suppose that the position function for an object in three dimensions is given by the equation $\mathbf{r}(t) = t\cos(t)\mathbf{i} + t\sin(t)\mathbf{j} + 3t\mathbf{k}$.

cone $z^2 = x^2 + y^2$



199. Show that the particle moves on a circular cone.

$x = t\cos t$

$y = t\sin t$

$z = 3t$

$$x^2 + y^2 = t^2(\cos^2 t + \sin^2 t)$$

$$= t^2 = \left(\frac{z}{3}\right)^2$$

$$z^2 = 9(x^2 + y^2) \quad \text{cone}$$

200. Find the angle between the velocity and acceleration vectors when $t = 1.5$.

$r(t) = \langle t\cos t, t\sin t, 3t \rangle$

$v(t) = \langle 1.\cos t + t(-\sin t), 1.\sin t + t\cos t, 3 \rangle$

$a(t) = \langle -\sin t + 1(-\sin t) + t(-\cos t), \cos t + 1.\cos t + t(-\sin t), 0 \rangle$

$a(t) = \langle -2\sin t - t\cos t, 2\cos t - t\sin t, 0 \rangle$

$a(1.5) = \langle -2.1, -1.4, 0 \rangle$

$v(1.5) = \langle -1.4, 1.1, 3 \rangle$

$$\cos\theta = \frac{a(1.5) \cdot v(1.5)}{\|a(1.5)\| \, \|v(1.5)\|}$$

Module 3
Differential Calculus

$z = f(x, y)$

Definition

A **function of two variables** $z = f(x, y)$ maps each ordered pair (x, y) in a subset D of the real plane $\mathbb{R}^2$ to a unique real number z. The set D is called the *domain* of the function. The *range* of f is the set of all real numbers z that has at least one ordered pair $(x, y) \in D$ such that $f(x, y) = z$ as shown in the following figure.

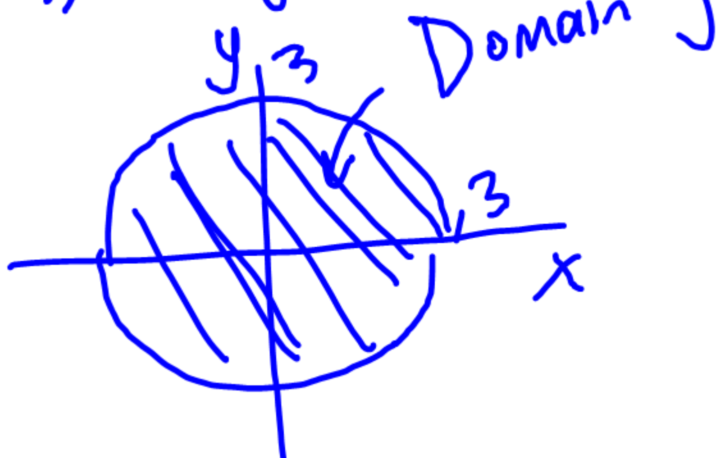
$z = 3x + 5y + 2$ $\qquad 3x + 5y - z + 2 = 0 \qquad Ax + By + Cz + D = 0$

$z = f(x, y) = 3x + 5y + 2$ $\qquad$ Plane $\qquad N = \langle A, B, C \rangle$

$z = g(x, y) = \sqrt{9 - x^2 - y^2}$ $\qquad$ Range is $\mathbb{R}$ $\qquad$ Domain is $\mathbb{R}^2$ defined for any x, y.

$9 - x^2 - y^2 \geq 0$

$9 \geq x^2 + y^2$



$z = \sqrt{9 - x^2 - y^2}$ semisphere

$x^2 + y^2 + z^2 = 3^2$ sphere of radius 3

Range is $[0, 3]$

$0 \leq z \leq 3$

4.1 Find the domain and range of the function $f(x, y) = \sqrt{36 - 9x^2 - 9y^2}$.

$$= \sqrt{36 - 9(x^2+y^2)}$$

upper half of the ellipsoid.

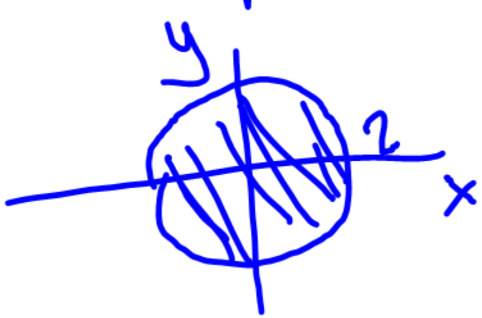
$z \geq 0$

$$36 - 9(x^2+y^2) \geq 0$$

$$36 \geq 9(x^2+y^2)$$

$$4 \geq x^2+y^2$$

D: circle of radius 2.



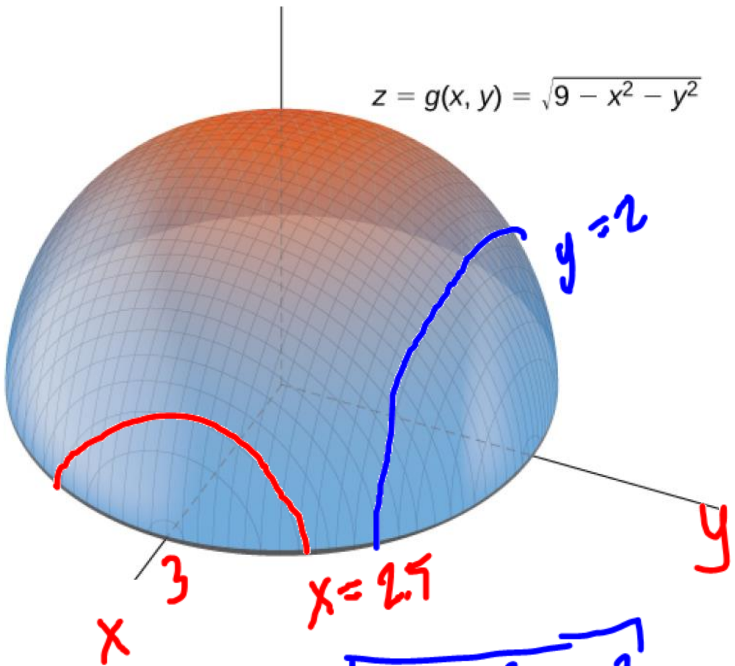
$0 \leq z \leq 6$ Range is $[0,6]$

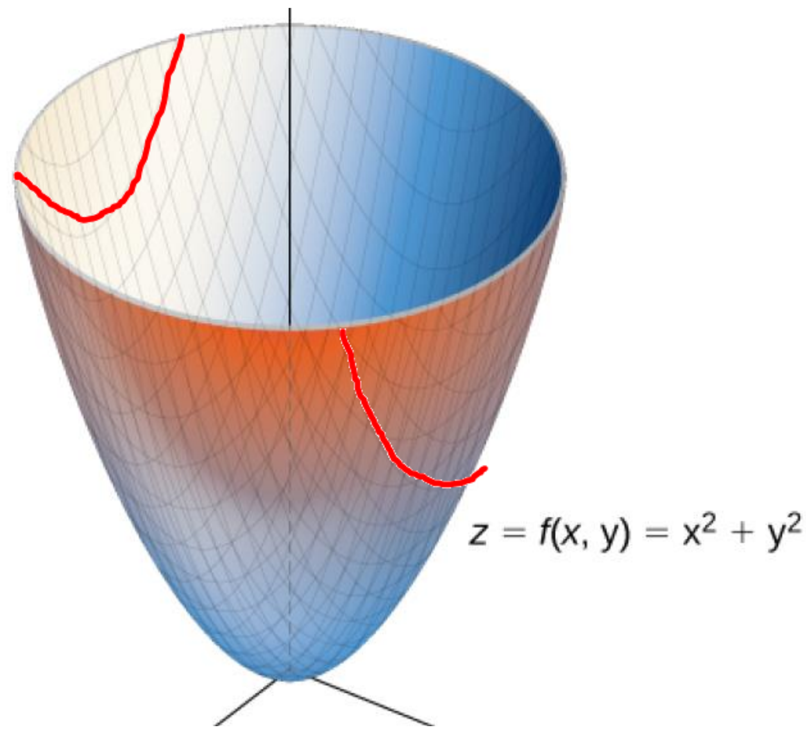
$$z^2 = 36 - 9x^2 + 9y^2$$

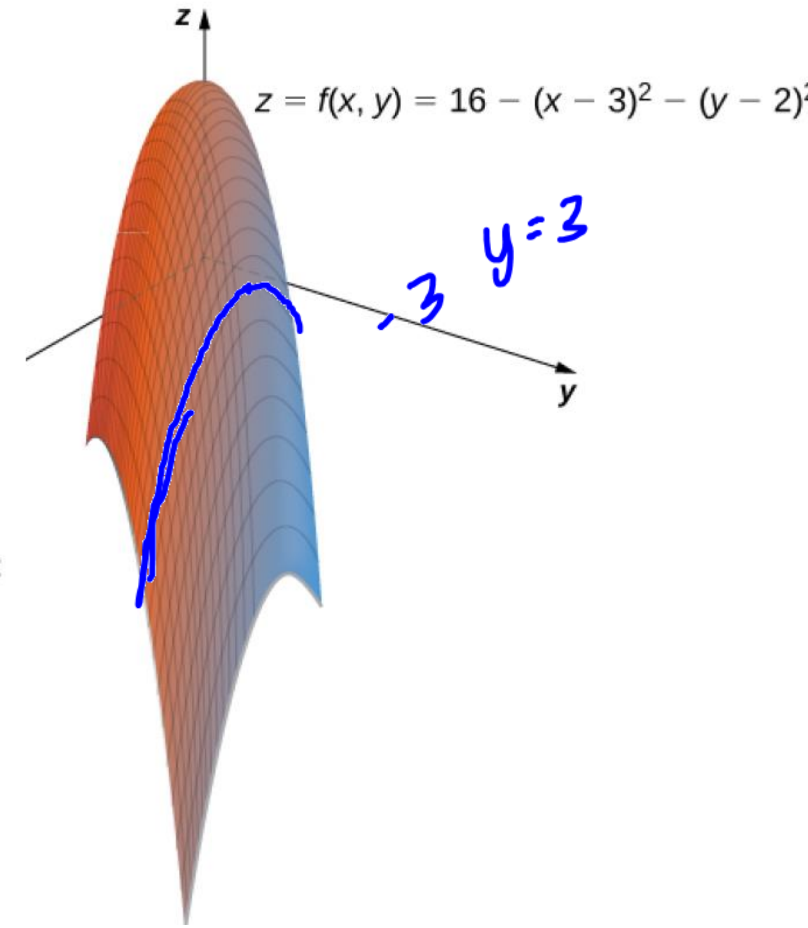
$$z^2 + 9x^2 + 9y^2 = 6^2 \quad \text{ellipsoid}$$

$$\frac{z^2}{6^2} + \frac{x^2}{2^2} + \frac{y^2}{2^2} = 1$$

The graph of a function $z = (x, y)$ of two variables is called a **surface**.







$z = \sqrt{9 - x^2 - 2^2}$

$z = \sqrt{5 - x^2}$ semicircle.

$z^2 + x^2 = 5$ circle

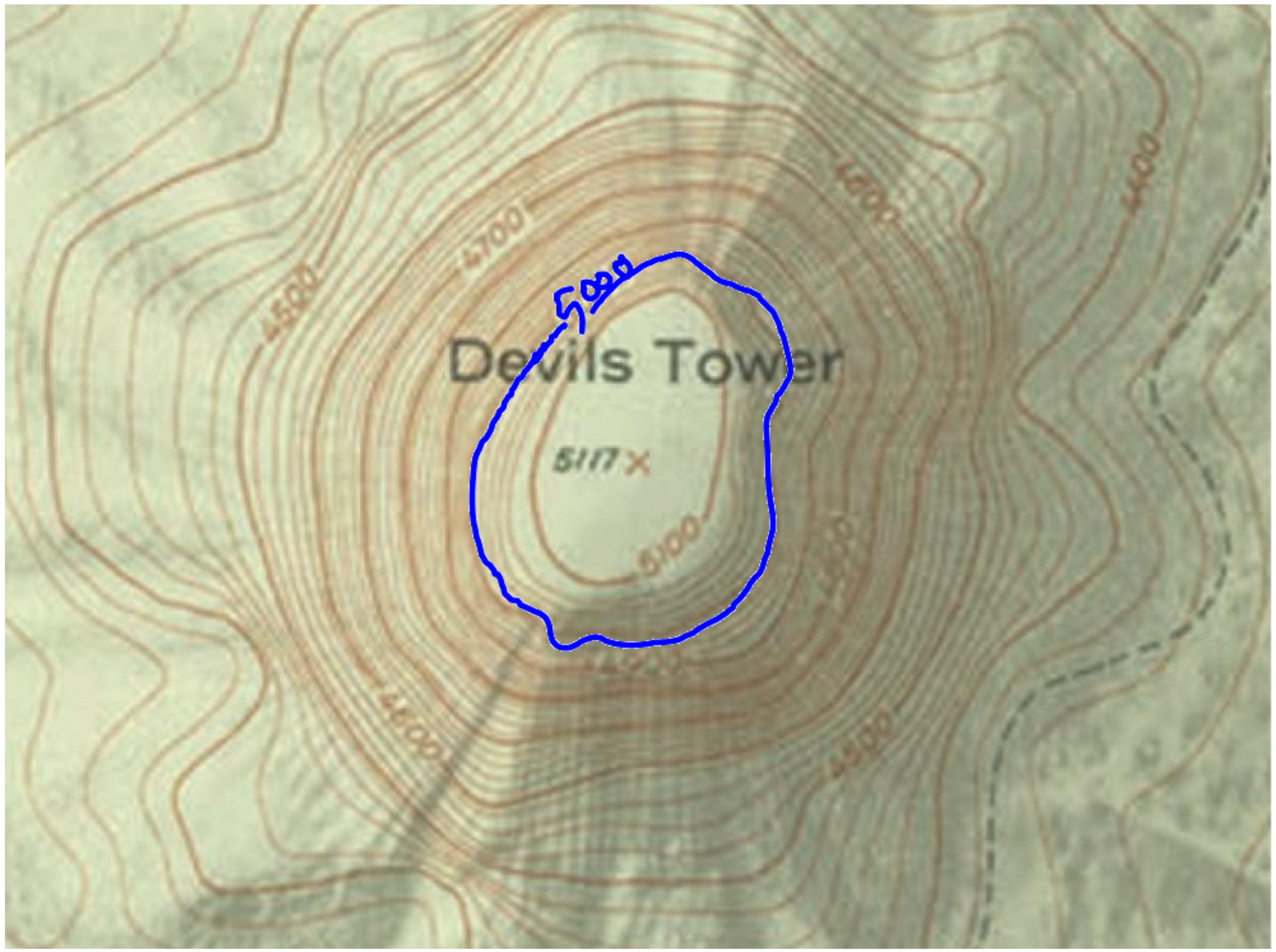
Definition

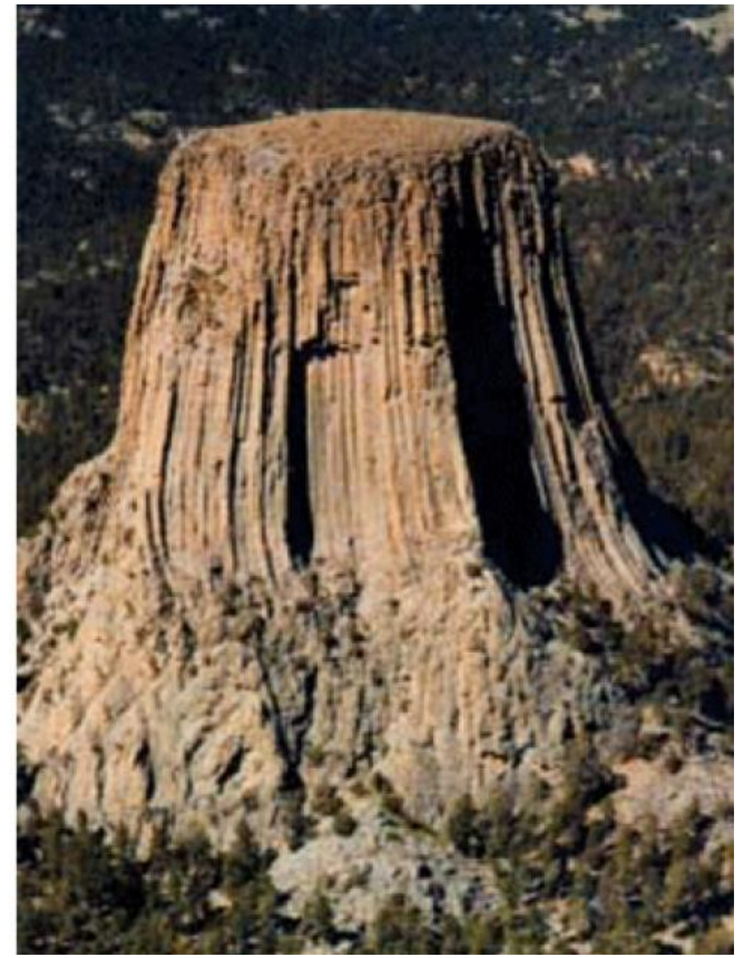
Given a function $f(x, y)$ and a number c in the range of f, a **level curve of a function of two variables** for the value c is defined to be the set of points satisfying the equation $f(x, y) = c$.

$z = f(x,y)$

fix z

A graph of the various level curves of a function is called a **contour map**.





4.2 Find and graph the level curve of the function $g(x, y) = x^2 + y^2 - 6x + 2y$ corresponding to $c = 15$.

paraboloid

$$15 = x^2 + y^2 - 6x + 2y$$

$$15 = x^2 - 6x + 9 - 9 + y^2 + 2y + 1 - 1$$

$$15 = (x-3)^2 + (y+1)^2 - 10$$

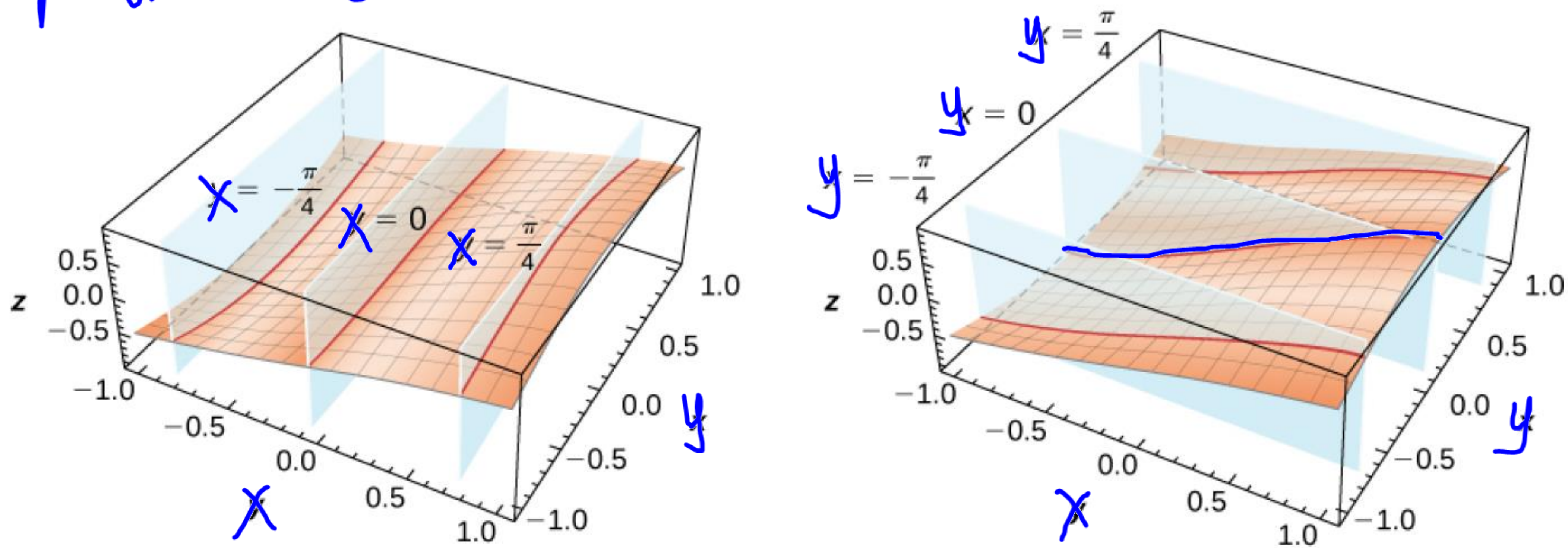
$$25 = (x-3)^2 + (y+1)^2$$ cirle of radius 5 centered at $(3, -1, 15)$

Definition

Consider a function $z = f(x, y)$ with domain $D \subseteq \mathbb{R}^2$. A **vertical trace** of the function can be either the set of points that solves the equation $f(a, y) = z$ for a given constant $x = a$ or $f(x, b) = z$ for a given constant $y = b$.

$z = f(0, y) = \sin 0 \sin y = 0$

$f(x, y) = \sin x \cos y$



Traces in the *xz*-planes

Traces in the *yz*-planes

4.3 Determine the equation of the vertical trace of the function $g(x, y) = -x^2 - y^2 + 2x + 4y - 1$ corresponding to $y = 3$, and describe its graph.

$z = g(x,3) = -x^2 - 9 + 2x + 12 - 1 = -x^2 + 2x + 2$

$z = -(x-1)^2 + 3$ parabola

Domains for Functions of Three Variables

Find the domain of $g(x, y, t) = \dfrac{\sqrt{2t - 4}}{x^2 - y^2}$

$x^2 - y^2 \neq 0$ means $y \neq \mp x$

$2t - 4 \geqslant 0$, $t \geqslant 2$

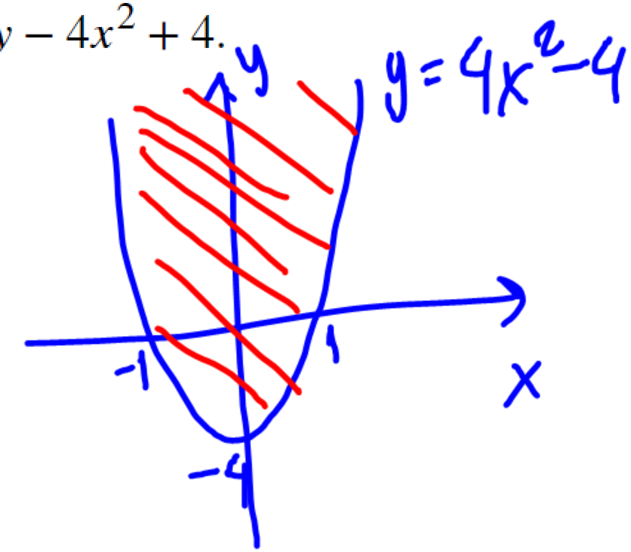
(x, y, t) such that $y \neq \mp x$, $t \geqslant 2$

4.4 Find the domain of the function $h(x, y, t) = (3t - 6)\sqrt{y - 4x^2 + 4}$.

No restriction for t.

$y - 4x^2 + 4 \geqslant 0$

$y \geqslant 4x^2 - 4$



Definition

Given a function $f(x, y, z)$ and a number c in the range of f, a **level surface of a function of three variables** is defined to be the set of points satisfying the equation $f(x, y, z) = c$.

4.5 Find the equation of the level surface of the function

$$g(x, y, z) = x^2 + y^2 + z^2 - 2x + 4y - 6z = 2$$

corresponding to $c = 2$, and describe the surface, if possible.

$$(x-1)^2 + (y+2)^2 + (z-3)^2 - 1 - 4 - 9 = 2$$

$$(x-1)^2 + (y+2)^2 + (z-3)^2 = 4^2$$ Sphere of radius 4

centered at $(1, -2, 3)$

For the following exercises, find the domain of the function.

5. $V(x, y) = 4x^2 + y^2$

6. $f(x, y) = \sqrt{x^2 + y^2 - 4}$

Find the range of the functions.

11. $g(x, y) = \sqrt{16 - 4x^2 - y^2}$ $0 \leq z \leq 4$ range is $[0, 4]$

12. $V(x, y) = 4x^2 + y^2$

$z = 4x^2 + y^2$ $0 \leq z \ldots$

elliptic paraboloid

range is $[0, \infty)$

Sketch the following by finding the level curves. Verify the graph using technology.

43. $f(x, y) = 2 - \sqrt{x^2 + y^2}$

Sketch the following by finding the level curves. Verify the graph using technology.

45. $z = \cos\sqrt{x^2 + y^2}$