Lines and planes in space quadric surfaces

Finding the Angle between Two Planes

$n = \langle a, b, c \rangle$ $\quad ax + by + cz + d = 0$

dot product

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \, \|\mathbf{n}_2\|}.$$

$0 \leq \theta \leq \frac{\pi}{2}$

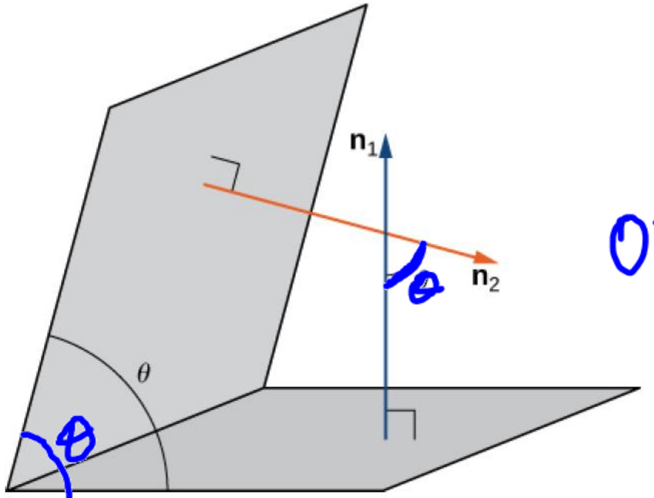


Figure 2.72 The angle between two planes has the same measure as the angle between the normal vectors for the planes.

2.50 Find the measure of the angle between planes $x + y - z = 3$ and $3x - y + 3z = 5$. Give the answer in radians and round to two decimal places.

$n_1 = \langle 1, 1, -1 \rangle \qquad n_2 = \langle 3, -1, 3 \rangle$

$\|n_1\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$

$\|n_2\| = \sqrt{3^2 + 1^2 + 3^2} = \sqrt{19}$

$\cos\theta = \dfrac{|n_1 \cdot n_2|}{\|n_1\|\|n_2\|} = \dfrac{|3 - 1 - 3|}{\sqrt{3}\sqrt{19}} = \dfrac{\sqrt{57}}{57} \approx 0.13$

$\theta = \cos^{-1}\left(\dfrac{\sqrt{57}}{57}\right) \cong 1.44$ radion

82.39 degree

Theorem 2.14: Distance from a Point to a Plane

Let $P(x_0, y_0, z_0)$ be a point. The distance from P to plane $ax + by + cz + k = 0$ is given by

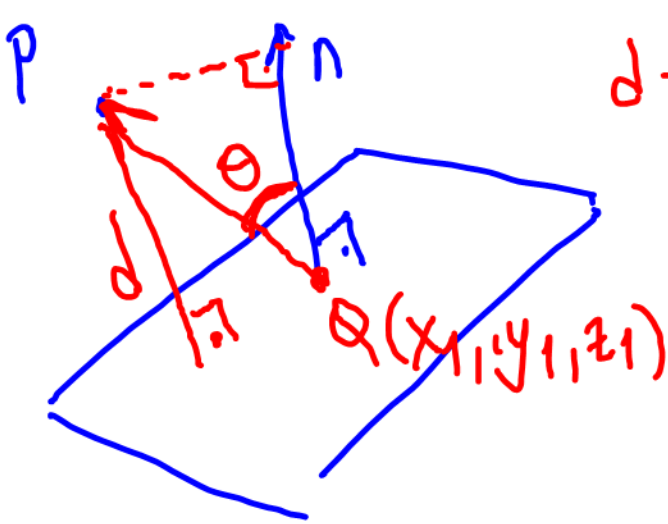
$$d = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}.$$

Q satisfies plane equation.

$ax_1 + by_1 + cz_1 + k = 0$

$\langle x_0 - x_1, y_0 - y_1, z_0 - z_1 \rangle \cdot \langle a, b, c \rangle$

$$d = \frac{\overrightarrow{QP} \cdot \mathbf{n}}{\| \mathbf{n} \|} = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|ax_0 + by_0 + cz_0 + k|}{\sqrt{a^2 + b^2 + c^2}}.$$



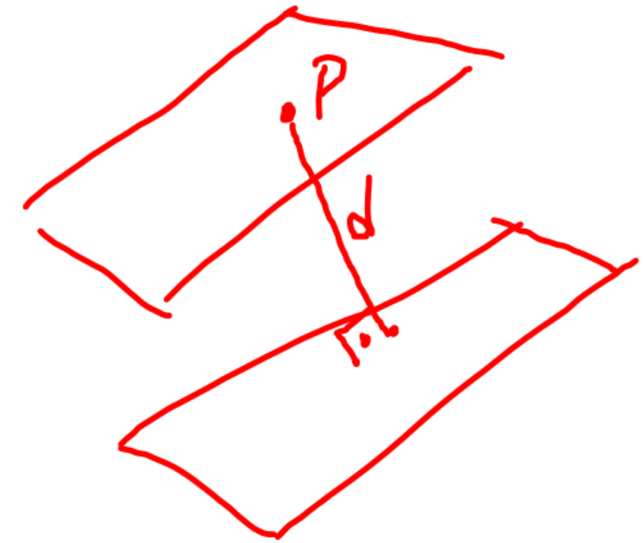
$d = \operatorname{comp}_{\vec{n}} \overrightarrow{QP} = \|QP\| \cos\theta$

$\overrightarrow{QP} \cdot \vec{n} = \|QP\| \|n\| \cos\theta$

2.51 Find the distance between parallel planes $5x - 2y + z = 6$ and $5x - 2y + z = -3$.

$x = 0 = y$ $\quad n = \langle 5, -2, 1 \rangle$

Let $P(0,0,6)$ on the first plane



$$d = \frac{|5\cdot 0 - 2\cdot 0 + 6 + 3|}{\sqrt{5^2 + 2^2 + 1}} = \frac{9}{\sqrt{30}} = \frac{3\sqrt{30}}{10}$$

254. Find the distance between point $A(4, 2, 5)$ and the line of parametric equations $x = -1 - t$, $y = -t$, $z = 2$, $t \in \mathbb{R}$.

$$\langle x, y, z \rangle = \langle -1, 0, 2 \rangle + t\langle -1, -1, 0 \rangle$$

$$P = (-1, 0, 2) \qquad \overrightarrow{PA} = \langle 5, 2, 3 \rangle$$

$$\vec{v} = \langle -1, -1, 0 \rangle$$

$$\overrightarrow{PA} \times \vec{v} = \begin{vmatrix} i & j & k \\ 5 & 2 & 3 \\ -1 & -1 & 0 \end{vmatrix} = i\,3 - j\,3 + k(-3)$$

$$d = \frac{\| \overrightarrow{PA} \times \vec{v} \|}{\| \vec{v} \|}$$

$$= \frac{\| \langle 3, -3, -3 \rangle \|}{\sqrt{1^2 + 1^2}}$$

$$= \frac{\sqrt{3^2 + 3^2 + 3^2}}{\sqrt{2}} = \frac{3\sqrt{3}}{\sqrt{2}}$$

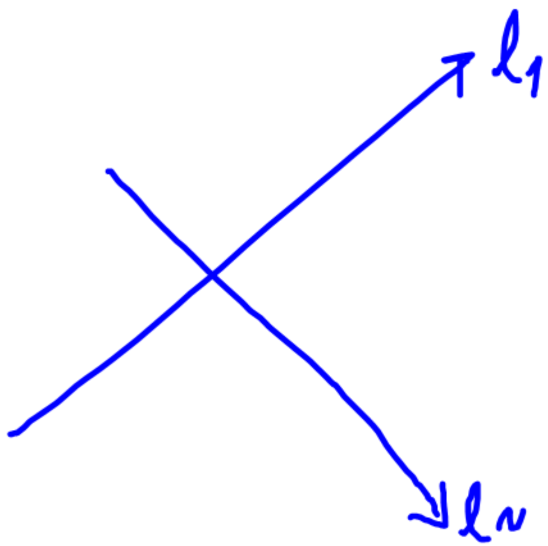
$$= \underline{\frac{3\sqrt{6}}{2}}$$

257. Show that the line passing through points $P(3,\ 1,\ 0)$ and $Q(1,\ 4,\ -3)$ is perpendicular to the line with equation $x = 3t,\ y = 3 + 8t,\ z = -7 + 6t,\quad t \in \mathbb{R}$.

$V_1 = \overrightarrow{PQ} = \langle -2, 3, -3 \rangle$

$V_2 = \langle 3, 8, 6 \rangle$

$V_1 \cdot V_2 = -2(3) + 3 \times 8 + (-3)6 = 0.$



$l_1 : \langle 3, 1, 0 \rangle + s \langle -2, 3, -3 \rangle$

$x = 3 - 2s = 3t$

$y = 1 + 3s = 3 + 8t$

$z = -3s = -7 + 6t$

$3 - 2 \times \frac{34}{7} = 3 \times \frac{5}{14}$

$1 = -4 + 14t \qquad t = 5/14$

$1 + 3s = 3 + 8 \times \frac{5}{14}$

$s = $ [illegible] $=$ [illegible]

$1 + 3s = 3 + \frac{4 \times 5}{7}$

$s = \frac{34}{7}$

Quadric Surfaces

$z^2 + y^2 = 1$

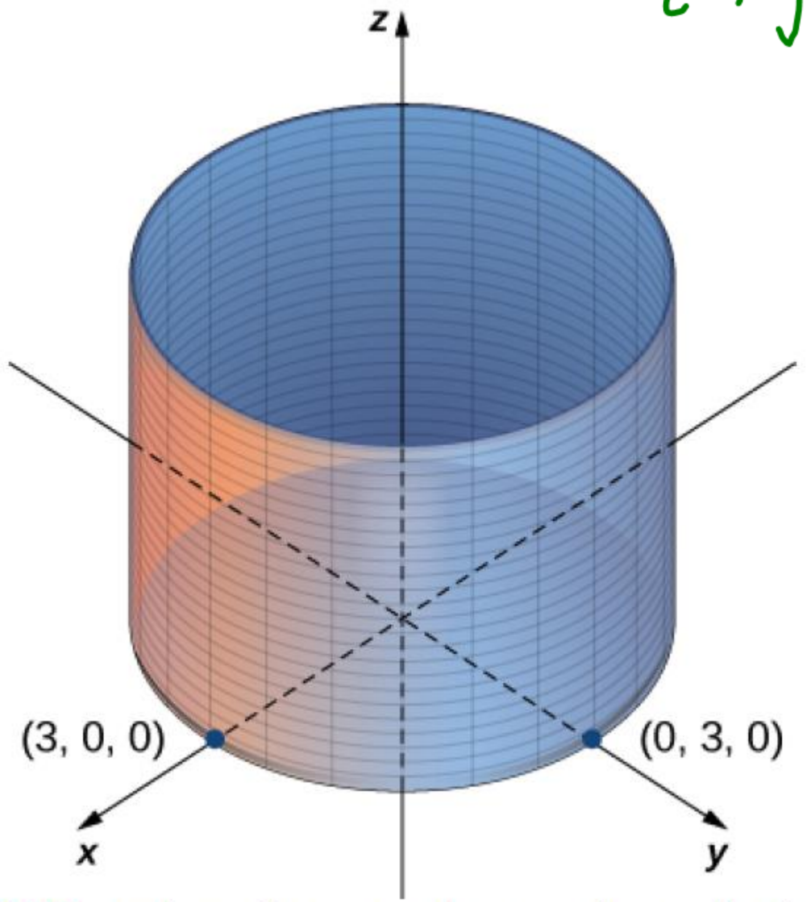


Figure 2.75 In three-dimensional space, the graph of equation $x^2 + y^2 = 9$ is a cylinder with radius 3 centered on the z-axis. It continues indefinitely in the positive and negative directions.

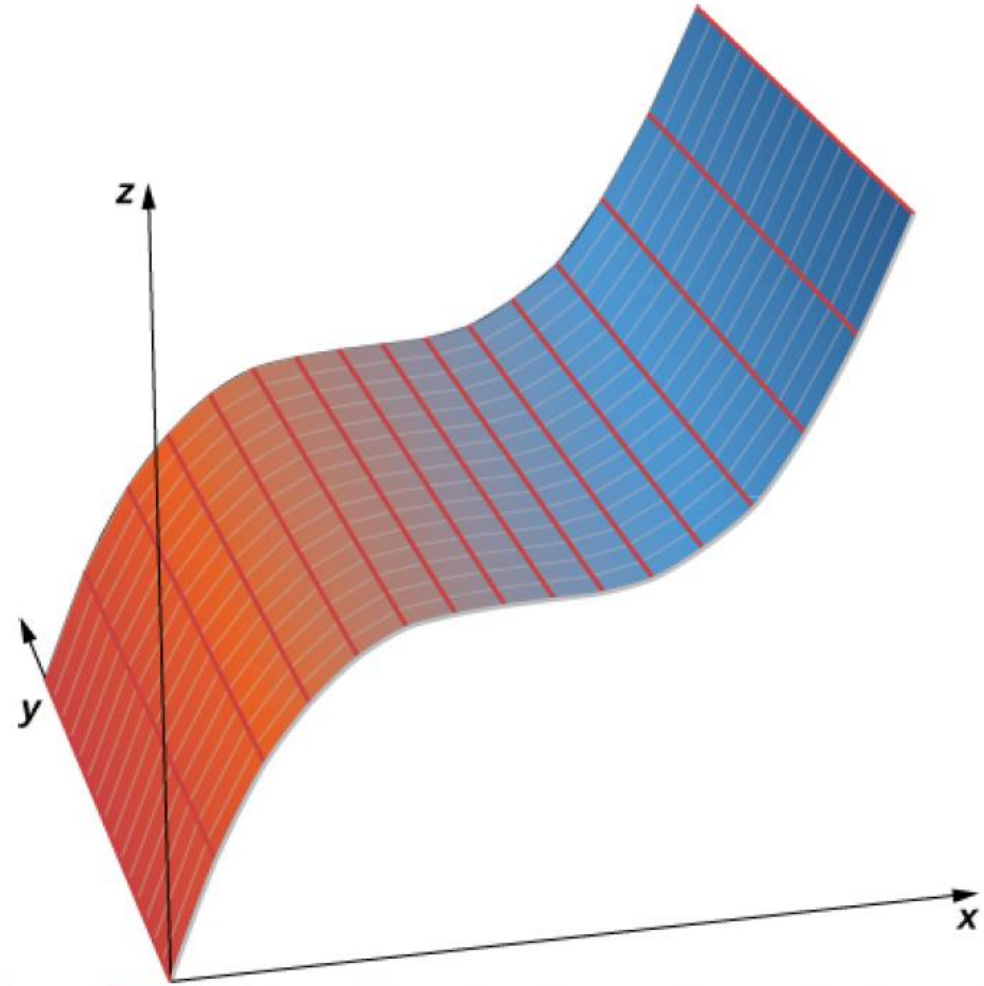
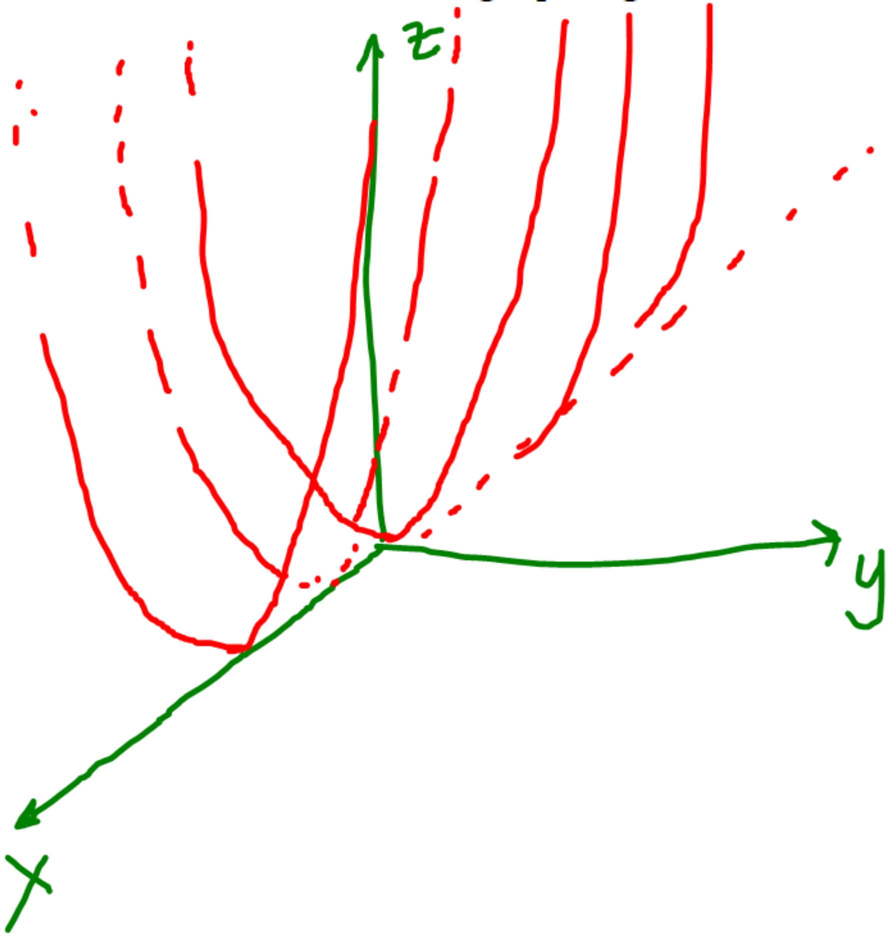


Figure 2.76 In three-dimensional space, the graph of equation $z = x^3$ is a cylinder, or a cylindrical surface with rulings parallel to the y-axis.

2.52 Sketch or use a graphing tool to view the graph of the cylindrical surface defined by equation $z = y^2$.



Definition

The **traces** of a surface are the cross-sections created when the surface intersects a plane parallel to one of the coordinate planes.

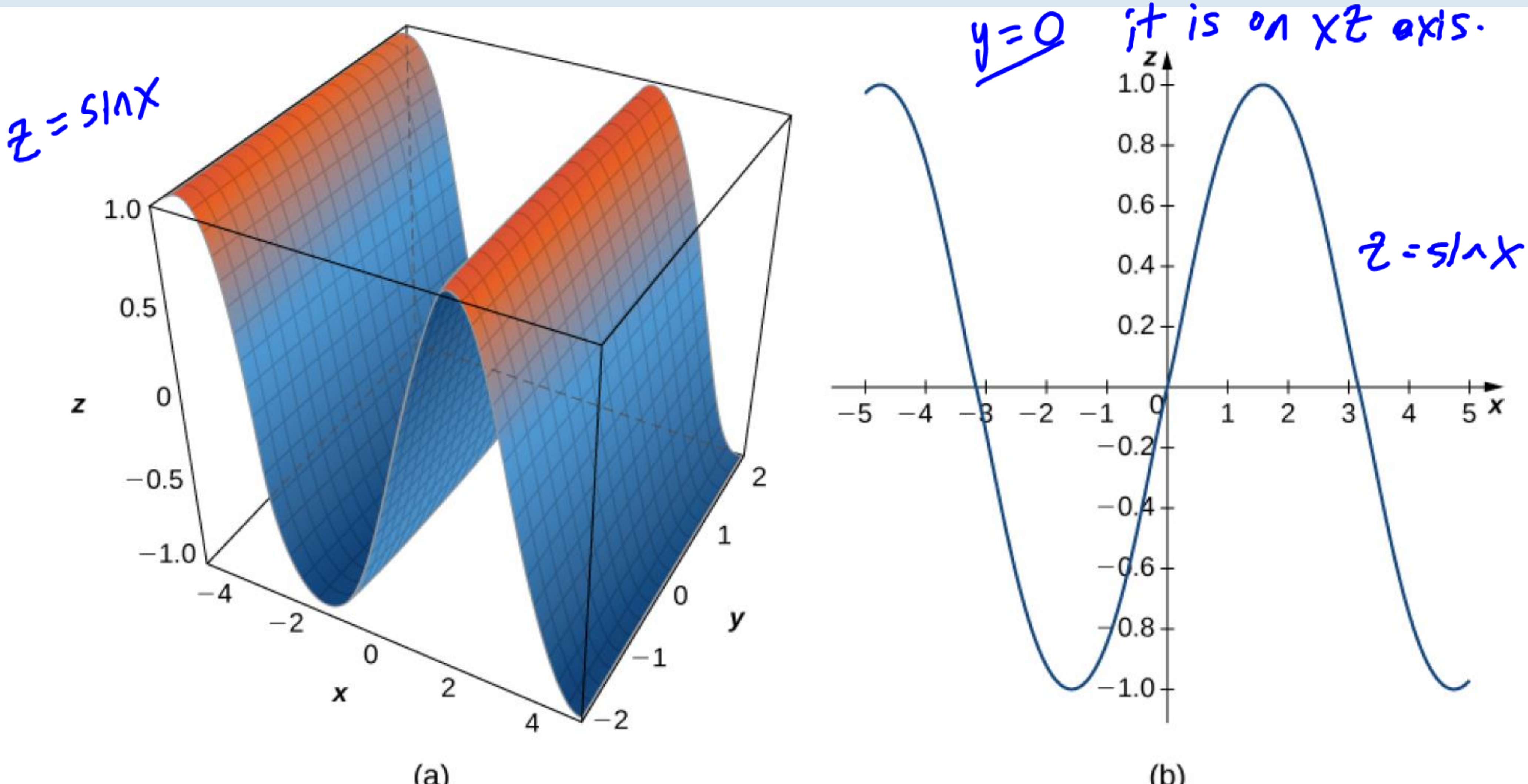


Figure 2.80 (a) This is one view of the graph of equation $z = \sin x$. (b) To find the trace of the graph in the xz-plane, set $y = 0$. The trace is simply a two-dimensional sine wave.

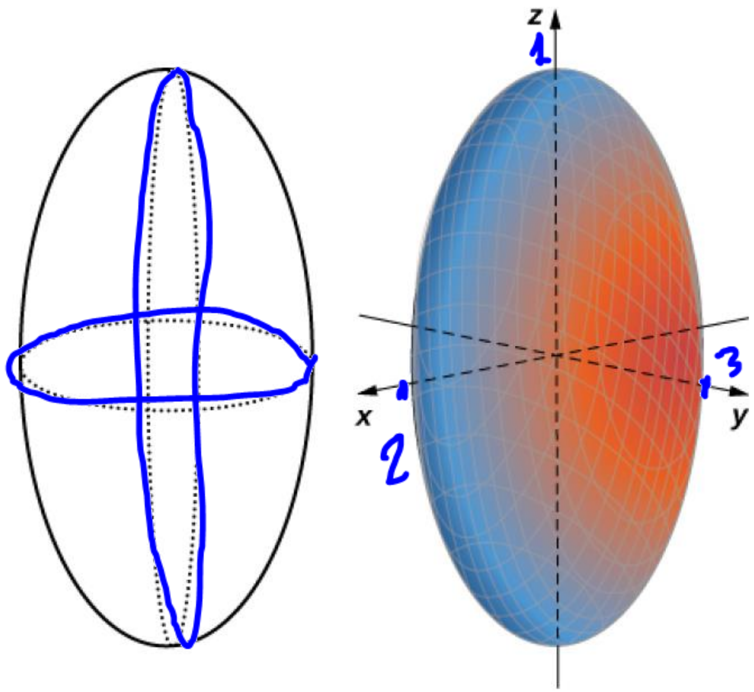
Definition

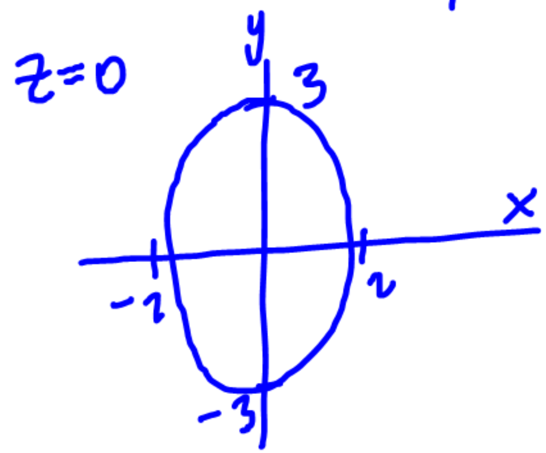
Quadric surfaces are the graphs of equations that can be expressed in the form

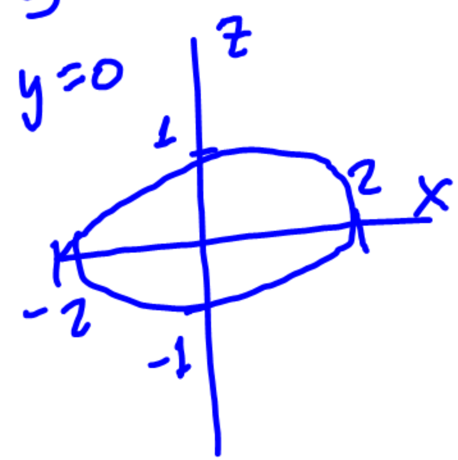
$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0.$$

An **ellipsoid** is a surface described by an equation of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$





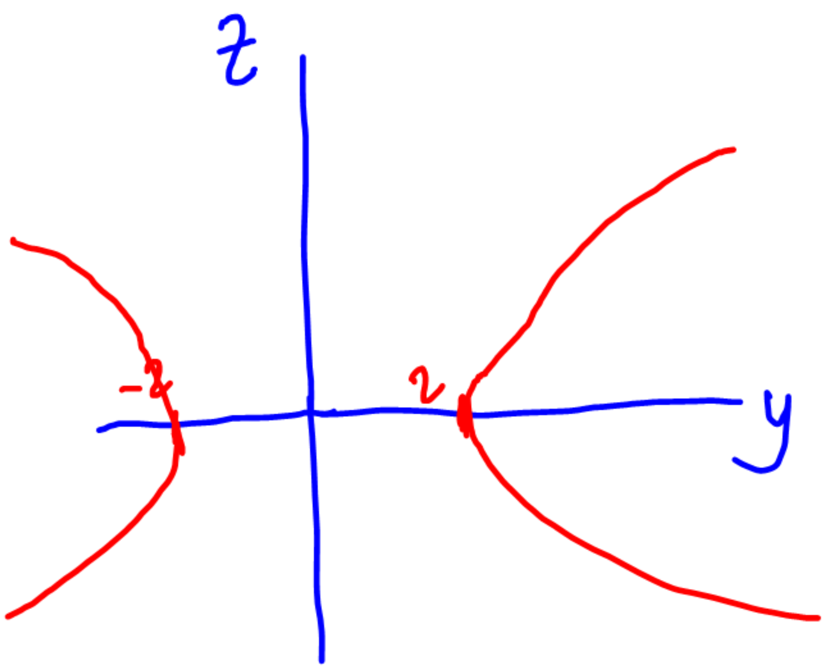


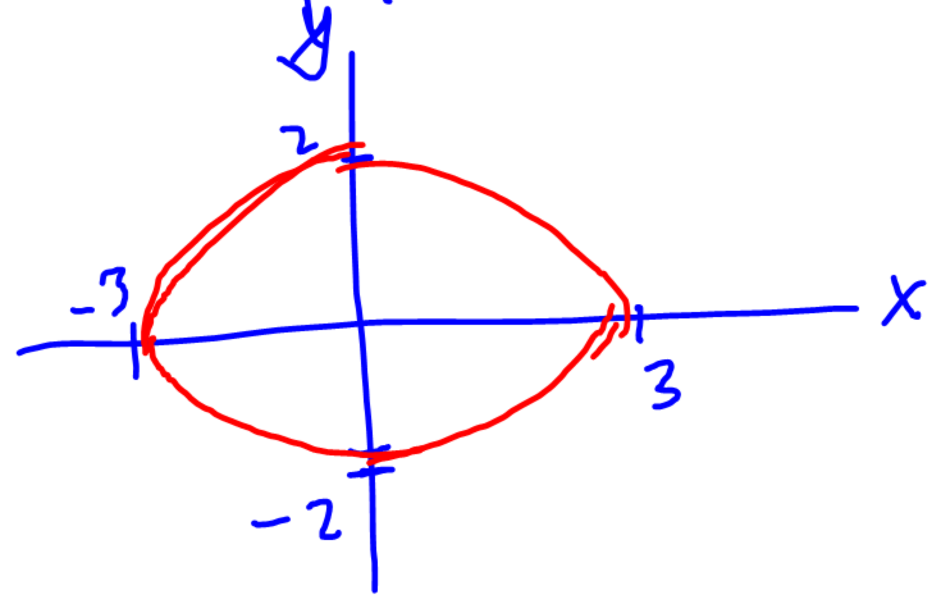
2.53 A hyperboloid of one sheet is any surface that can be described with an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$. Describe the traces of the hyperboloid of one sheet given by equation $\frac{x^2}{3^2}+\frac{y^2}{2^2}-\frac{z^2}{5^2}=1$.

$x=0 \quad \frac{y^2}{2^2}-\frac{z^2}{5^2}=1$

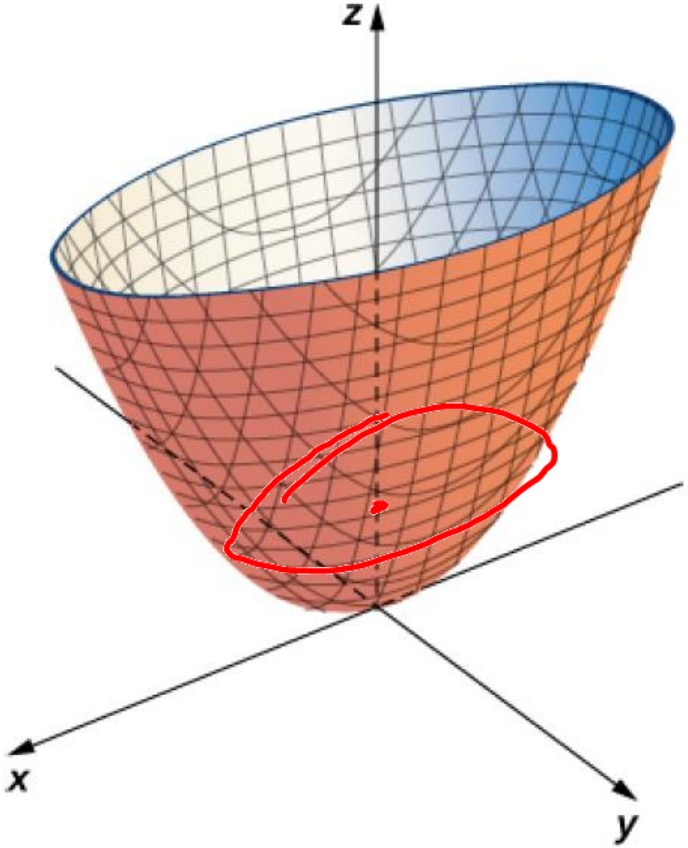
$y=0$ hyperbola

$z=0 \quad \frac{x^2}{9}+\frac{y^2}{4}=1$ ellipse

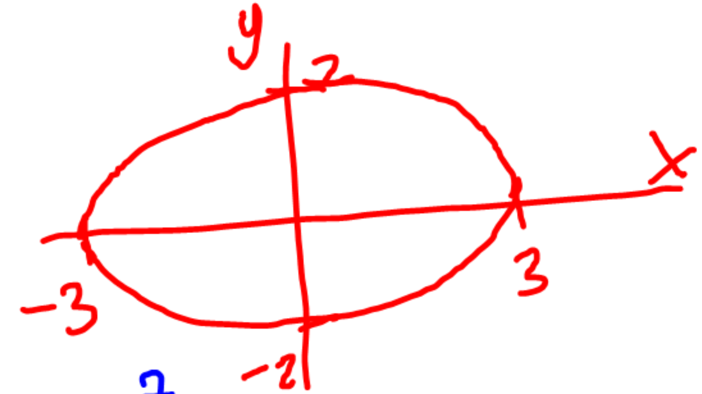




$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$ **elliptic paraboloid**



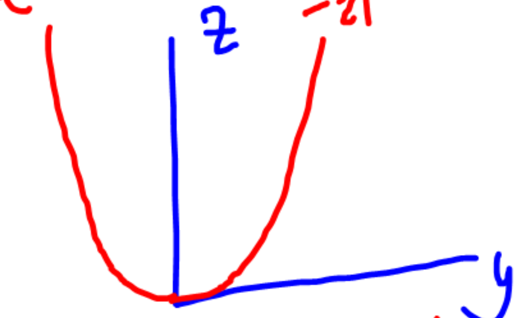
$$\frac{x^2}{9}+\frac{y^2}{4}=z$$



$z=1$ ellipse

$z=2$ larger ellipse

$x=0$, $z=\frac{y^2}{4}$



Similary for any other x-value.

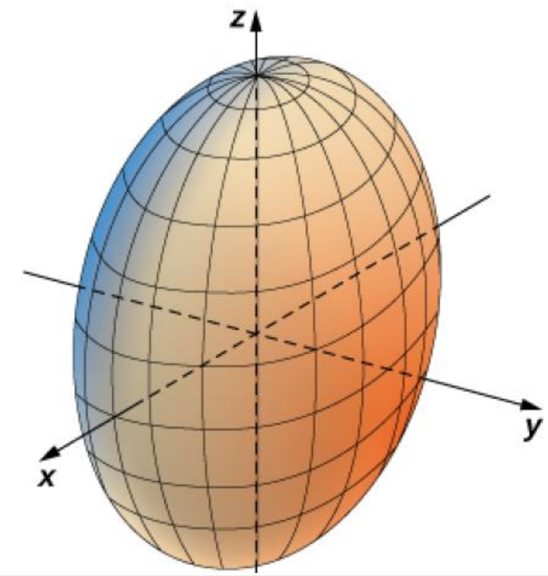
or y-value. (xz)

Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: an ellipse
In plane $x = r$: an ellipse

If $a = b = c$, then this surface is a sphere.

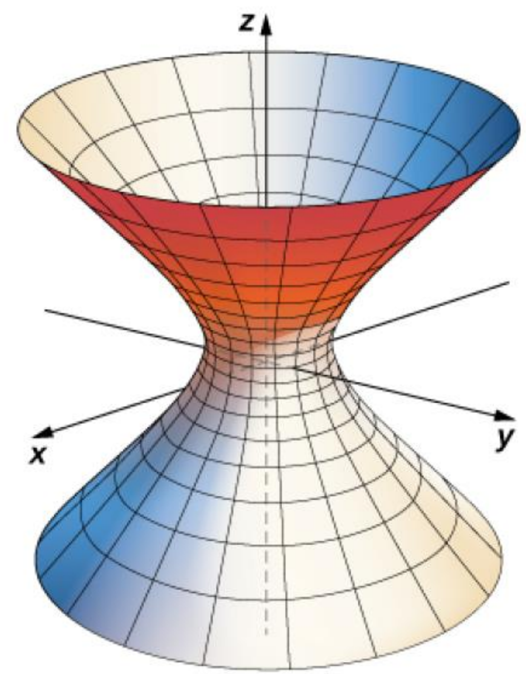


Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.

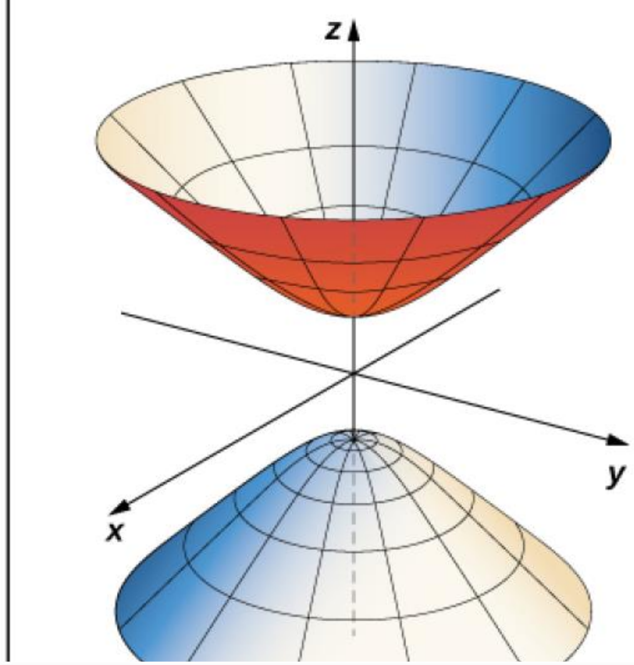


Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Traces
In plane $z = p$: an ellipse or the empty set (no trace)
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola

In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.

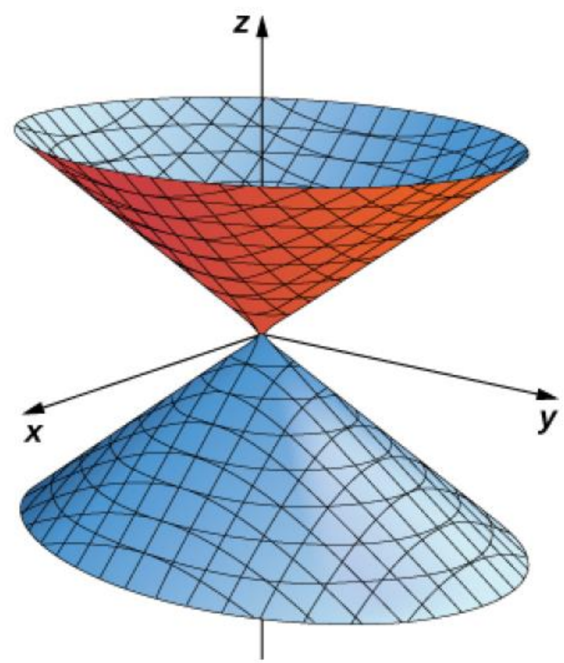


Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

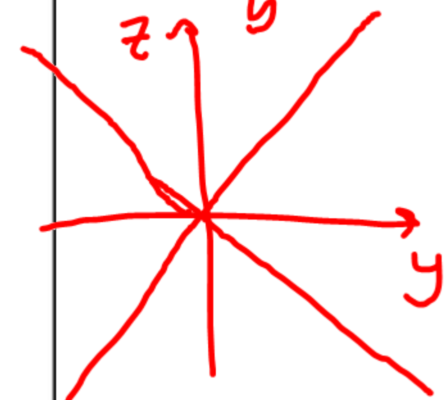
Traces
In plane $z = p$: an ellipse
In plane $y = q$: a hyperbola
In plane $x = r$: a hyperbola
In the xz – plane: a pair of lines that intersect at the origin
In the yz – plane: a pair of lines that intersect at the origin

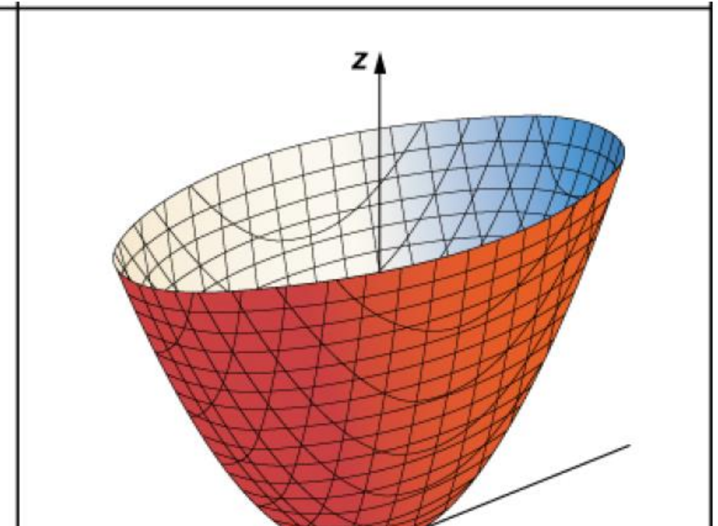
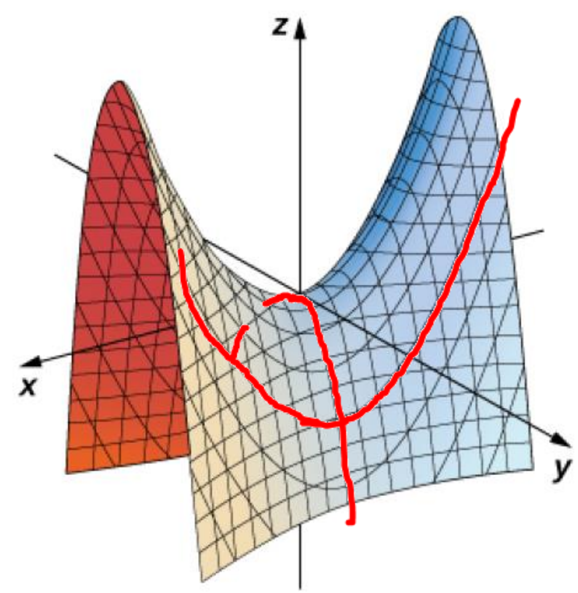
The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.



$x=0$

$z = \mp \frac{cy}{b}$



Elliptic Paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ *Traces* In plane $z = p$: an ellipse In plane $y = q$: a parabola In plane $x = r$: a parabola The axis of the surface corresponds to the linear variable.	
Hyperbolic Paraboloid $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ *Traces* In plane $z = p$: a hyperbola In plane $y = q$: a parabola In plane $x = r$: a parabola The axis of the surface corresponds to the linear variable.	

$y = 3$

$z = \frac{x^2}{4} - \frac{9}{9}$

$x = 0$

$z = -\frac{y^2}{9}$

2.54 Identify the surface represented by equation $9x^2 + y^2 - z^2 + 2z - 10 = 0$.

$$9x^2 + y^2 - (z^2 - 2z + 1) + 1 - 10 = 0$$

$$x^2 + \frac{y^2}{9} - \frac{(z-1)^2}{9} = 1$$

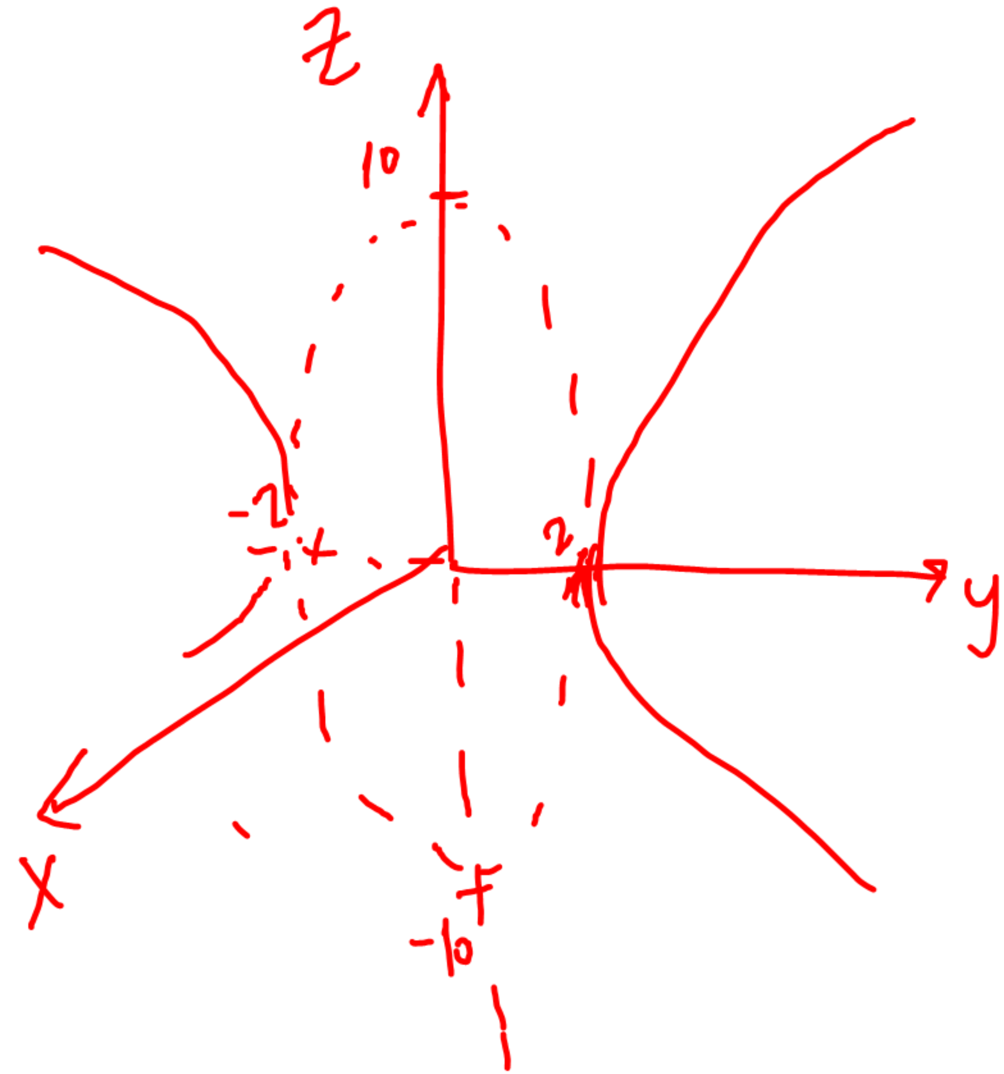
hyperboloid of one sheet.

For the following exercises, rewrite the given equation of the quadric surface in standard form. Identify the surface.

320. $-4x^2 + 25y^2 + z^2 = 100$

$$-\frac{x^2}{25} + \frac{y^2}{4} + \frac{z^2}{100} = 1$$

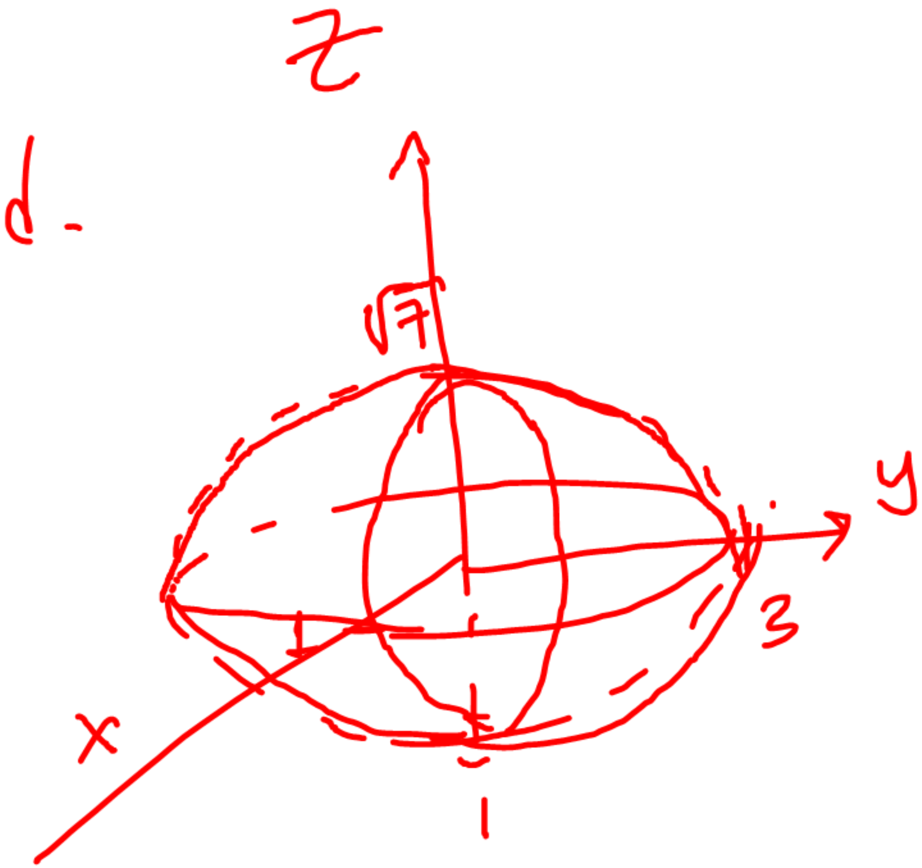
hyperboloid of one sheet.



For the following exercises, rewrite the given equation of the quadric surface in standard form. Identify the surface.

326. $63x^2 + 7y^2 + 9z^2 - 63 = 0$

$$x^2 + \frac{y^2}{9} + \frac{z^2}{7} = 1$$ ellipsoid.



For the following exercises, match the given quadric surface with its corresponding equation in standard form.

a. $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{12} = 1$

b. $\frac{x^2}{4} - \frac{y^2}{9} - \frac{z^2}{12} = 1$

c. $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{12} = 1$

d. $z = 4x^2 + 3y^2$

e. $z = 4x^2 - y^2$

f. $4x^2 + y^2 - z^2 = 0$

313. Hyperboloid of two sheets

314. Ellipsoid

315. Elliptic paraboloid

316. Hyperbolic paraboloid

317. Hyperboloid of one sheet

318. Elliptic cone

For the following exercises, the equation of a quadric surface is given.

a. Use the method of completing the square to write the equation in standard form.

b. Identify the surface.

$$(x+3)^2+2(z-2)^2=16$$

$$(x^2+6x+9)-9+2(z^2-4z+4)-8+1=0$$

339. $x^2+2z^2+6x-8z+1=0$

cylindrical ellipse

$$\frac{(x+3)^2}{16}+\frac{(z-2)^2}{8}=1$$

344. $x^2-y^2+z^2-12z+2x+37=0$

$$(x+1)^2-y^2+(z-6)^2-1-36+37=0$$

$$(x+1)^2-y^2+(z-6)^2=0$$

elliptic cone

347. Determine the intersection points of elliptic cone $x^2 - y^2 - z^2 = 0$ with the line of symmetric equations $\frac{x-1}{2} = \frac{y+1}{3} = z = t$

$$x = 1 + 2t$$
$$y = -1 + 3t$$
$$z = t$$

$$(1+2t)^2 - (-1+3t)^2 - t^2 = 0$$

$$4t^2 + 4t + \cancel{1} - 9t^2 + 6t - \cancel{1} - t^2 = 0$$

$$-6t^2 + 10t = 0$$

$$-2t(3t-5) = 0$$

$t = 0$ OR $t = \frac{5}{3}$

$(1, -1, 0)$ $\left(\frac{13}{3}, 4, \frac{5}{3}\right)$