Exercise 1. *Find* $|\vec{u}+\vec{v}|$ *if* $\vec{u}=\langle 2,-1,1\rangle$ *and* $\vec{v}=\langle -1,3,13\rangle$.

Solution. $\vec{u}+\vec{v}=\langle 2-1,-1+3,1+13\rangle=\langle 1,2,14\rangle$ *then* $|\vec{u}+\vec{v}|=\sqrt{1^2+2^2+14^2}=\sqrt{201}$.

Exercise 2. *Determine whether the vectors* $\vec{u}=\langle 1,2,2\rangle$, $\vec{v}=\left\langle \sqrt{2},1,-1\right\rangle$ *are orthogonal, parallel or neither. If neither, also find the angle between two vectors.*

Solution. $\vec{u}\cdot\vec{v}=\sqrt{2}+2-2=\sqrt{2}$, *the vectors are neither orthogonal nor parallel. Let* θ *be the angle between the vectors. Then,*

$$\cos\theta=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|}=\frac{\sqrt{2}}{\sqrt{1+2^2+2^2}\sqrt{2+1+1}}=\frac{\sqrt{2}}{3\times 2}=\frac{\sqrt{2}}{6}.$$

Therefore, $\theta=\arccos\dfrac{\sqrt{2}}{6}$.

Exercise 3. *Find* $\cos\widehat{ABC}$ *if* $A(1,4)$, $B(2,2)$ *and* $C(3,5)$. *Find the measure of the angle* $\widehat{ABC}$.

Solution. $\widehat{ABC}$ *is the angle between the vectors* $\vec{BA}$ *and* $\vec{BC}$, *which are* $\vec{BA}=\vec{A}-\vec{B}=\langle -1,2\rangle$ *and* $\vec{BC}=\vec{C}-\vec{B}=\langle 1,3\rangle$. *Therefore,*

$$\cos\widehat{ABC}=\frac{\vec{BA}\cdot\vec{BC}}{\left|\vec{BA}\right|\left|\vec{BC}\right|}=\frac{-1+6}{\sqrt{5}\sqrt{10}}=\frac{5}{5\sqrt{2}}=\frac{\sqrt{2}}{2}.$$

Hence, $m\left(\widehat{ABC}\right)=45^\circ$.

Exercise 4. *Find* $\cos\widehat{BCA}$ *if* $A(1,4)$, $B(2,2)$ *and* $C(3,5)$. *Find the measure of the angle* $\widehat{ABC}$.

Solution. $\widehat{BCA}$ *is the angle between the vectors* $\vec{CB}$ *and* $\vec{CA}$, *which are* $\vec{CB}=\langle -1,-3\rangle$ *and* $\vec{CA}=\langle -2,-1\rangle$. *Therefore,*

$$\cos\widehat{BCA}=\frac{\vec{CB}\cdot\vec{CA}}{\left|\vec{CB}\right|\left|\vec{CA}\right|}=\frac{2+3}{\sqrt{10}\sqrt{5}}=\frac{5}{5\sqrt{2}}=\frac{\sqrt{2}}{2}.$$

Hence, $m\left(\widehat{BCA}\right)=45^\circ$. *In fact, the triangle* $\overset{\triangle}{ABC}$ *is a right triangle. We may verify that* $\vec{AB}\cdot\vec{AC}=0$.

Exercise 5. *Find the vector projection* $proj_{\vec{v}}\vec{u}$ *if* $\vec{u}=\langle 2,-1\rangle$ *and* $\vec{v}=\langle 1,3\rangle$.

Solution.

$$proj_{\vec{v}}\vec{u}=\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}=\frac{-1}{10}\vec{v}=\left\langle\frac{-1}{10},\frac{3}{10}\right\rangle$$

Exercise 6. *Find the vector projection* $proj_{\vec{v}}\vec{u}$ *if* $\vec{u}=\langle 0,1,2\rangle$ *and* $\vec{v}=\left\langle 1,1,\sqrt{2}\right\rangle$.

Solution.

$$proj_{\vec{v}}\vec{u}=\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}=\frac{0+1+2\sqrt{2}}{4}\vec{v}=\left\langle\frac{1+2\sqrt{2}}{4},\frac{1+2\sqrt{2}}{4},\frac{\sqrt{2}+4}{4}\right\rangle$$

Exercise 7. *Find* $|\vec{u}\times\vec{v}|$ *if* $|\vec{u}|=5, |\vec{v}|=6$, *and the angle between* $\vec{u}$ *and* $\vec{v}$ *is* 30°.

Solution. $|\vec{u}\times\vec{v}|=|\vec{u}|\,|\vec{v}|\sin\theta$ *where* θ *is the angle between the vectors. Therefore,*

$$|\vec{u}\times\vec{v}|=5\times 6\sin 30^\circ=15.$$

Exercise 8. *Find symmetric equations of the line through the point* $P_0(-2,1,3)$ *and parallel to the line* $x=2+t,\ y=-1+5t,\ z=4t$.

Solution. *The line* $x=2+t,\ y=-1+5t,\ z=4t$ *has vector equation* $\langle 2,-1,0\rangle+t\langle 1,5,4\rangle$ *and direction vector* $\vec{u}=\langle 1,5,4\rangle$. *Therefore, the line parallel to this through* P_0 *will have*

$$\begin{aligned}\textit{vector equation } \langle x,y,z\rangle &= \langle -2,1,3\rangle+t\langle 1,5,4\rangle,\\ \textit{parametric equation } x &= -2+t,\ y=1+5t,\ z=3+4t \textit{ and}\\ \textit{symmetric equations } \frac{x+2}{1} &= \frac{y-1}{5}=\frac{z-3}{4}=t.\end{aligned}$$

Exercise 9. *Find a vector equation of the line through the points $A(2,4,3)$ and $B(1,2,-1)$. Also give parametric equations for the line. Where does the line intersect xz−plane?*

Solution. *$\vec{BA} = \vec{A} - \vec{B} = \langle 1,2,4\rangle$ is a direction vector for the line. Therefore, it has*

$$\begin{aligned} \textit{vector equation } \langle x,y,z\rangle &= \langle 2,4,3\rangle + t\langle 1,2,4\rangle, \\ \textit{parametric equation } x &= 2+t,\ y = 4+2t,\ z = 3+4t \end{aligned}$$

The points on xz-plane has y-coordinate 0. So let $y = 0 = 4+2t$ implies the line intersects with xz-plane when $t = -2$ which corresponds the point $(0,0,-5)$.

Exercise 10. *Determine whether the planes $2x + y - z = 1$ and $x + y + 3z = 2$ are parallel, perpendicular or neither. If neither, also find the angle between two planes.*

Solution. *The angle between the planes is equal to the angle between the normal vectors. The normal vectors are $N_1 = \langle 2,1,-1\rangle$ and $N_2 = \langle 1,1,3\rangle$. The dot product $N_1 \cdot N_2 = 2+1-3 = 0$, which implies $N_1 \perp N_2$. Therefore, the planes are perpendicular to each other.*

Exercise 11. *Determine whether the planes $\sqrt{2}x + y + z = 1$ and $\sqrt{2}x - y + z = 5$ are parallel, perpendicular or neither. If neither, also find the angle between two planes.*

Solution. *Let θ be the angle between the planes, which is also the angle between their normal vectors $N_1 = \langle \sqrt{2},1,1\rangle$ and $N_2 = \langle \sqrt{2},-1,1\rangle$. Then*

$$\cos\theta = \frac{N_1 \cdot N_2}{|N_1|\,|N_2|} = \frac{2-1+1}{\sqrt{2+1+1}\sqrt{2+1+1}} = \frac{2}{4} = \frac{1}{2}.$$

Hence, $\theta = \frac{\pi}{3}$. The planes are neither parallel nor perpendicular. The angle between the planes is $60°$.

Exercise 12. *Find the distance from the point $P(4,5,6)$ to the plane $2x - y + z = 6$.*

Solution. *The distance between the point $P_0(x_0,y_0,z_0)$ and the plane with eqation $Ax+By+Cz+D = 0$ is given by the formula*

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\left|\sqrt{A^2+B^2+C^2}\right|}$$

Therefore, the distance between the given point and the plane is

$$\frac{|2\times 4 - 5 + 6 - 6|}{\left|\sqrt{2^2+1^2+1^2}\right|} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$$

Exercise 13. *Let $\mathcal{P}$ be the plane containing the point $(2,1,1)$ and perpendicular to x-axis. Find the intersection of the plane $\mathcal{P}$ with the sphere of radius 3 centered at the origin?*

Solution. *Any plane perpendicular to x-axis will have $\boldsymbol{i} = \langle 1,0,0\rangle$ as its normal vector. Therefore, it will have equation $x+D = 0$. Since it passes through the point $(2,1,1)$ we get $D = -2$. In other words, the plane has equation $x = 2$. Now the sphere of radius 3 has equation*

$$\begin{aligned} x^2+y^2+z^2 &= 3^2, \textit{ substitute } x = 2 \textit{ to find the intersection,} \\ \textit{we obtain } y^2+z^2 &= 5 \textit{ and } x = 2. \end{aligned}$$

That is the circle of radius $\sqrt{5}$ centered at $(2,0,0)$ and perpendicular to x-axis.

Exercise 14. *Let $\mathcal{P}$ be the plane with equation $x + 2y + z = 10$ and l be the line through the points $A(1,0,-1)$ and $B(2,1,1)$. Find the point of intersection if they intersect.*

Solution. *The line l has equation $\langle x,y,z\rangle = \vec{A} + t\vec{AB} = \langle 1,0,-1\rangle + t\langle 1,1,2\rangle$. A parametric form of the line equation is*

$$x = 1+t,\quad y = t,\quad z = -1+2t$$

The line intersects with the plane. Since if they were parallel the direction vector $\langle 1,1,2\rangle$ would be perpendicular to the normal vector $\langle 1,2,1\rangle$.

$$\langle 1,1,2\rangle \cdot \langle 1,2,1\rangle = 1+2+2 = 5 \neq 0.$$

We may find the intersection point by common solution. Substitute the parametric equations into the equation of the plane

$$\begin{aligned} 1+t+2t+1+2t &= 10, \\ 5t+2 &= 10, \\ t &= 8/5 \end{aligned}$$

So, the intersection point is $x = 1+8/5,\ y = 8/5,\ z = 1+16/5$, that is $\left(\frac{13}{5}, \frac{8}{5}, \frac{21}{5}\right)$.

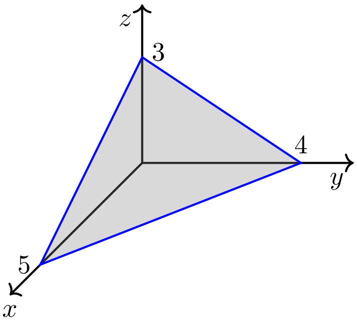
Exercise 15. *By using triple product, find the volume of the parallelepiped determined by the vectors* $\vec{u} = \langle 0, 2, 1 \rangle$, $\vec{v} = \langle -1, 3, 0 \rangle$ *and* $\vec{w} = \langle 2, 1, -1 \rangle$.

Solution. *The volume of the parallelepiped is given by the triple product (the absolute value of the determinant)*

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 0 & 2 & 1 \\ -1 & 3 & 0 \\ 2 & 1 & -1 \end{vmatrix} = 0 \begin{vmatrix} 3 & 0 \\ 1 & -1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 0 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} = 0 - 2 - 7 = -9, \text{ or by Sarrus rule.}$$

As a result, we see that the volume is 9 *unit*^{3}.

Exercise 16. *Find an equation of the plane containing the given triangle.*



Solution. *The cross product of any non-parallel pair of vectors in the plane will give us a normal vector of the plane. Let* $A(5,0,0)$, $B(0,4,0)$ *and* $C(0,0,3)$ *be the vertices of the given triangle. Then* $\vec{AB} = \langle -5, 4, 0 \rangle$ *and* $\vec{AC} = \langle -5, 0, 3 \rangle$. *Therefore*

$$\vec{N} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -5 & 4 & 0 \\ -5 & 0 & 3 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} 4 & 0 \\ 0 & 3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} -5 & 0 \\ -5 & 3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} -5 & 4 \\ -5 & 0 \end{vmatrix} = \langle 12, 15, 20 \rangle$$

The equation of the plane is $12x + 15y + 20y + D = 0$ *for some* D. *We obtain* $60 + D = 0$ *when we substitute the coordinates of* A. *Hence, the equation of the plane is* $12x + 15y + 20y - 60 = 0$.

Exercise 17. *Calculate the dot product* $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v})$ *if* $\vec{u} = \boldsymbol{i} + \boldsymbol{3j} + \boldsymbol{k}$ *and* $\vec{v} = 5\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$. *Is the angle between the vectors* $\vec{u} + \vec{v}$ *and* $\vec{u} - \vec{v}$ *obtuse or acute? Find the angle between* $\vec{u}$ *and* $\vec{v}$.

Solution. $\vec{u} + \vec{v} = \langle 6, 2, -1 \rangle$ *and* $\vec{u} - \vec{v} = \langle -4, 4, 3 \rangle$. *Therefore,*

$$(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \langle 6, 2, -1 \rangle \cdot \langle -4, 4, 3 \rangle = -24 + 8 - 3 = -19$$

The dot product is negative which implies that the cosine value of the angle between these vectors is negative. So, the angle between these is obtuse. On the other hand,

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \langle 1, 3, 1 \rangle \cdot \langle 5, -1, -2 \rangle = 5 - 3 - 2 = 0, \\ &\quad \textit{which implies } \vec{u} \perp \vec{v}\textit{; the vectors are orthogonal.} \end{aligned}$$

Exercise 18. *Calculate the cross product* $(\vec{u} + 2\vec{v}) \times (2\vec{u} - \vec{v})$ *if* $\vec{u} = \boldsymbol{j} + 2\boldsymbol{k}$ *and* $\vec{v} = 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$.

Solution. $\vec{u} + 2\vec{v} = 4\boldsymbol{i} - \boldsymbol{j} + 4\,\boldsymbol{k} = \langle 4, -1, 4 \rangle$ *and* $2\vec{u} - \vec{v} = -2\boldsymbol{i} + 3\boldsymbol{j} + 3\boldsymbol{k} = \langle -2, 3, 3 \rangle$

$$\begin{aligned} (\vec{u} + 2\vec{v}) \times (2\vec{u} - \vec{v}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 4 & -1 & 4 \\ -2 & 3 & 3 \end{vmatrix} = \boldsymbol{i} \begin{vmatrix} -1 & 4 \\ 3 & 3 \end{vmatrix} - \boldsymbol{j} \begin{vmatrix} 4 & 4 \\ -2 & 3 \end{vmatrix} + \boldsymbol{k} \begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix} \\ &= \boldsymbol{i}\,(-15) - \boldsymbol{j}\,(20) + \boldsymbol{k}\,(10) = \langle -15, -20, 10 \rangle. \end{aligned}$$

Exercise 19. *Find the limit* $\lim_{t \to 2} \left\langle t^2, \dfrac{\sin(t-2)}{t^2 - 4}, e^t \right\rangle$.

Solution. $\lim_{t \to 2} t^2 = 4$, $\lim_{t \to 2} \dfrac{\sin(t-2)}{t^2 - 4} = \lim_{t \to 2} \dfrac{\sin(t-2)}{(t-2)(t+2)} = \dfrac{1}{4}$ *and* $\lim_{t \to 2} e^t = e^2$. *Therefore,*

$$\lim_{t \to 2} \left\langle t^2, \frac{\sin(t-2)}{t^2 - 4}, e^t \right\rangle = \left\langle 4, \frac{1}{4}, e^2 \right\rangle.$$

Exercise 20. *For the vector function* $\vec{r}(t) = \langle t^2, \cos t, e^{2t} \rangle$ *find the second order derivative when* $t = 0$. *In other words,* $\vec{r}\,''(0) = ?$

Solution. *The first order derivative of the function is* $\vec{r}\,'(t) = \langle 2t, -\sin t, 2e^{2t} \rangle$. *Then,*

$$\vec{r}\,''(t) = \langle 2, -\cos t, 4e^{2t} \rangle \text{ and } \vec{r}\,''(0) = \langle 2, -1, 4 \rangle .$$

Exercise 21. *Find the rate of change for vector function* $\vec{r}(t) = \langle \sin t, \cos t, \tan t \rangle$ *when* $t = \pi/6$.

Solution. *The rate of change for* $\vec{r}(t)$ *when* $t = \pi/6$ *is the first derivative at the given value* $\vec{r}\,'(\pi/6)$.

$$\vec{r}\,'(t) = \langle \cos t, -\sin t, \sec^2 t \rangle \text{ and } \vec{r}\,'(\pi/6) = \left\langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{4}{3} \right\rangle.$$

Exercise 22. *Determine whether the vector-valued function* $\vec{r}(t) = \left\langle \frac{1}{t+2}, \ln(t-2), t^2 \right\rangle$ *is continuous or not at* $t = 2$.

Solution. *The function is undefined at* $t = 2$, *since the* y*-component* $y(t) = \ln(t-2)$ *is undefined for* $t = \dot{2}$. *So, the function is not continuous at* $t = 2$.

Exercise 23. *Find the vector function* $\vec{r}(t)$ *if* $\vec{r}'(t) = \langle 2t, \cos t, e^t \rangle$ *and* $\vec{r}(0) = \langle 1, 2, 3 \rangle$.

Solution. $\vec{r}(t) = \int \vec{r}'(t)\,dt = \langle t^2 + c_1, \sin t + c_2, e^t + c_3 \rangle = \langle t^2, \sin t, e^t \rangle + \langle c_1, c_2, c_3 \rangle$. *We find the integration constant by*

$$\begin{aligned} \vec{r}(0) &= \langle 1,2,3 \rangle = \langle 0,0,1 \rangle + \langle c_1, c_2, c_3 \rangle, \text{ implies } \langle c_1, c_2, c_3 \rangle = \langle 1,2,2 \rangle, \\ \text{Therefore, } \vec{r}(t) &= \langle t^2 + 1, \sin t + 2, e^t + 2 \rangle . \end{aligned}$$

Exercise 24. *Evaluate the integral* $\int_0^1 \left(t\boldsymbol{i} + \frac{2t}{1+t^2}\boldsymbol{j} + e^t \boldsymbol{k} \right) dt.$

Solution.

$$\begin{aligned} \int_0^1 \left(t\boldsymbol{i} + \frac{2t}{1+t^2}\boldsymbol{j} + e^t\boldsymbol{k} \right) &= \left(\boldsymbol{i}\frac{t^2}{2} + \boldsymbol{j}\ln(1+t^2) + \boldsymbol{k}e^t \right)\Big|_{t=0}^{t=1} \\ &= \frac{\boldsymbol{i}}{2} + (\ln 2)\,\boldsymbol{j} + (e-1)\,\boldsymbol{k} = \left\langle \frac{1}{2}, \ln 2, e-1 \right\rangle. \end{aligned}$$

Exercise 25. *The velocity of an object is given by* $\vec{v}(t) = \left\langle 2t, \sin t, \frac{1}{t+1} \right\rangle$ *and* $\vec{r}(0) = \langle 1,1,1 \rangle$. *Find the position function* $\vec{r}(t)$.

Solution. *The velocity is the first order derivative of the position function;* $\vec{v}(t) = \vec{r}'(t)$. *Then,*

$$\begin{aligned} \vec{r}(t) &= \int \vec{r}'(t)\,dt = \int \vec{v}(t)\,dt = \langle t^2 + c_1, -\cos t + c_2, \ln(t+1) + c_3 \rangle, \\ \vec{r}(0) &= \langle c_1, c_2 - 1, c_3 \rangle = \langle 1,1,1 \rangle \text{ implies } c_1 = 1,\ c_2 = 2 \text{ and } c_3 = 1, \\ \text{Therefore, } \vec{r}(t) &= \langle t^2 + 1, -\cos t + 2, \ln(t+1) + 1 \rangle . \end{aligned}$$

Exercise 26. *Find the length of the curve* $\vec{r}(t) = \langle \sqrt{5}t, \cos 2t, -\sin 2t \rangle$ *from* $t = 0$ *to* $t = 2\pi$.

Solution. *The arclength is* $\int_0^{2\pi} |\vec{r}'(t)|\,dt,$

$$\begin{aligned} \vec{r}\,'(t) &= \langle \sqrt{5}, -2\sin 2t, -2\cos 2t \rangle \text{ and } |\vec{r}'(t)| = \sqrt{5 + 4(\sin^2 2t + \cos^2 2t)} = 3 \\ S &= \int_0^{2\pi} 3dt = 3t|_{t=0}^{t=2\pi} = 6\pi. \end{aligned}$$

Exercise 27. *Find the unit tangent vector* $\vec{T}(t)$ *to the curve* $\vec{r}(t) = \langle t, \cos t, \sin t \rangle$ *at* $t = \pi/3$.

Solution. $\vec{r}'(t) = \langle 1, -\sin t, \cos t \rangle$ *and* $|\vec{r}'(t)| = \sqrt{1 + \sin^2 t + \cos^2 t} = \sqrt{2}$. *The unit tangent vector is*

$$\begin{aligned} \vec{T}(t) &= \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle 1, -\sin t, \cos t \rangle}{\sqrt{2}}, \\ \vec{T}(\pi/3) &= \left\langle \frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} \right\rangle = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4} \right\rangle. \end{aligned}$$

Exercise 28. *Find the curvature of the function* $\vec{r}(t) = \langle t, t^2, 0 \rangle$ *at* $t = 1$.

Solution. *We have various formulas for the curvature. For this question, we are going to use*

$$\kappa = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

We have $\vec{r}\,'(t) = \langle 1, 2t, 0 \rangle$ *and* $\vec{r}\,''(t) = \langle 0, 2, 0 \rangle$, *which gives*

$$\begin{aligned} \vec{r}\,'(t) \times \vec{r}\,''(t) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2t & 0 \\ 0 & 2 & 0 \end{vmatrix} = 2\boldsymbol{k}, \\ \kappa &= \frac{2}{\sqrt{1+4t^2}^3} \text{ when } t = 1 \text{ we get } \frac{2}{5\sqrt{5}} = \frac{2\sqrt{5}}{25}. \end{aligned}$$