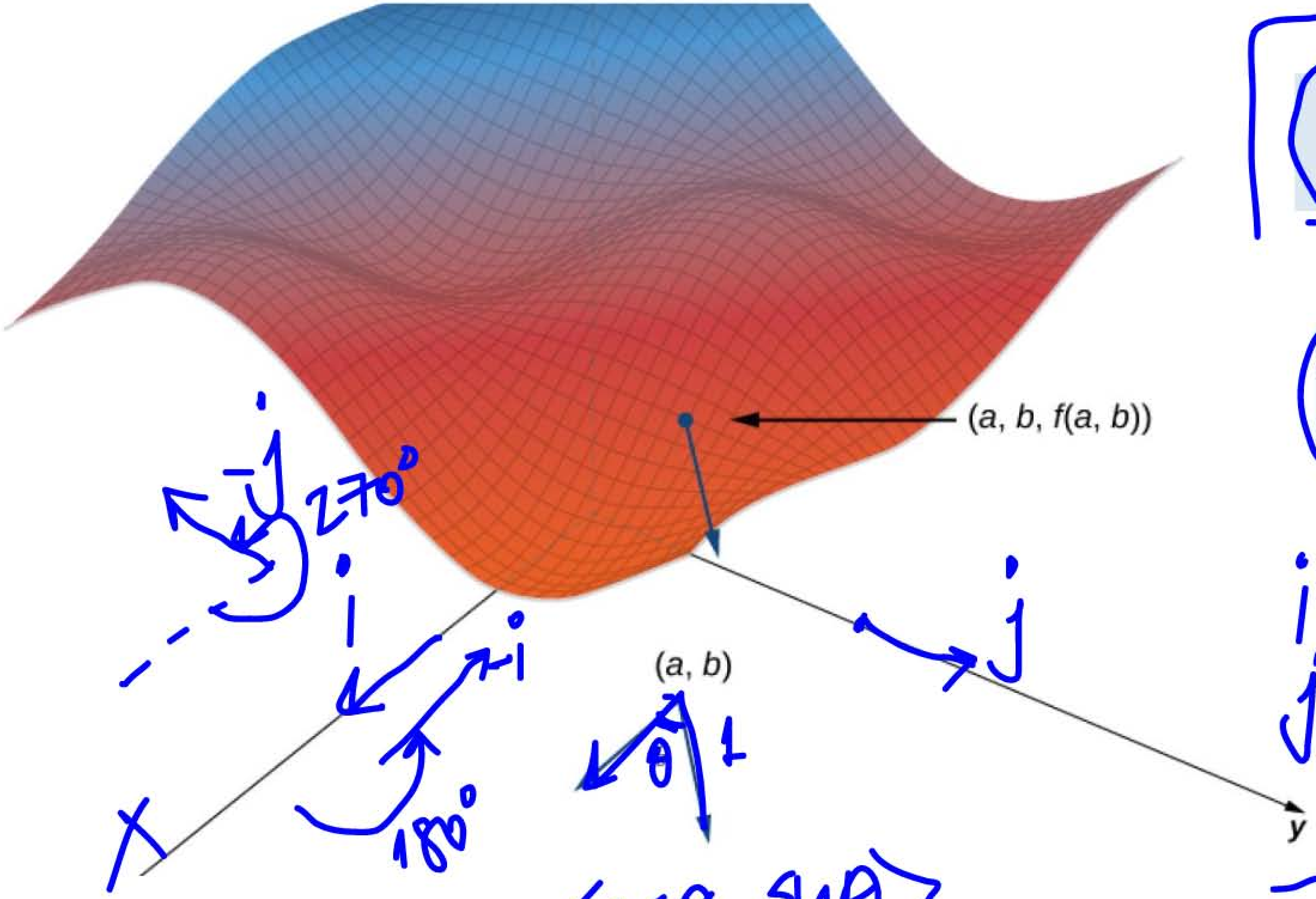
4.6 | Directional Derivatives and the Gradient



$\langle \cos\theta, \sin\theta \rangle$

$$\boxed{\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,\ y) - f(x,\ y)}{h}}$$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x,\ y+k) - f(x,\ y)}{k}$$

$i \to \theta = 0^\circ$

$j \to \theta = \frac{\pi}{2}$

$-j \to \theta = \frac{3\pi}{2}$

Definition

Suppose $z = f(x, y)$ is a function of two variables with a domain of D. Let $(a, b) \in D$ and define $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$. Then the **directional derivative** of f in the direction of u is given by

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a, b)}{h}, \tag{4.36}$$

provided the limit exists.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

combine → D_u directional derivative $u = i$ $\theta = 0^\circ$

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

→ " " $u = j$ $\theta = 90^\circ$

Theorem 4.12: Directional Derivative of a Function of Two Variables

Let $z = f(x, y)$ be a function of two variables x and y, and assume that f_x and f_y exist. Then the directional derivative of f in the direction of $\mathbf{u} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is given by

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \quad (4.37)$$

easier

$\langle f_x, f_y \rangle \cdot \langle -1, 0 \rangle$

264. $h(x, y) = e^x \sin y$, $P\left(1, \frac{\pi}{2}\right)$, $\mathbf{v} = -\mathbf{i}$

$\theta = 180^\circ = \pi$

$D_u h(x,y) = e^x \sin y \cdot \underbrace{\cos\pi}_{-1} + e^x \cos y \cdot \underbrace{\sin\pi}_{0}$

$= -e^x \sin y = -h_x$

$D_j = h_y$

$D_{-j} = -h_y$

For the following exercises, find the directional derivative of the function at point P in the direction of $\mathbf{v}$.

272. $f(x, y) = x^2 y$, $P(-5, 5)$, $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$

$u = \langle \cos\theta, \sin\theta \rangle$

$D_u f = \langle f_x, f_y \rangle \cdot \langle \cos\theta, \sin\theta \rangle$

$D_v f = \langle f_x, f_y \rangle \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle$

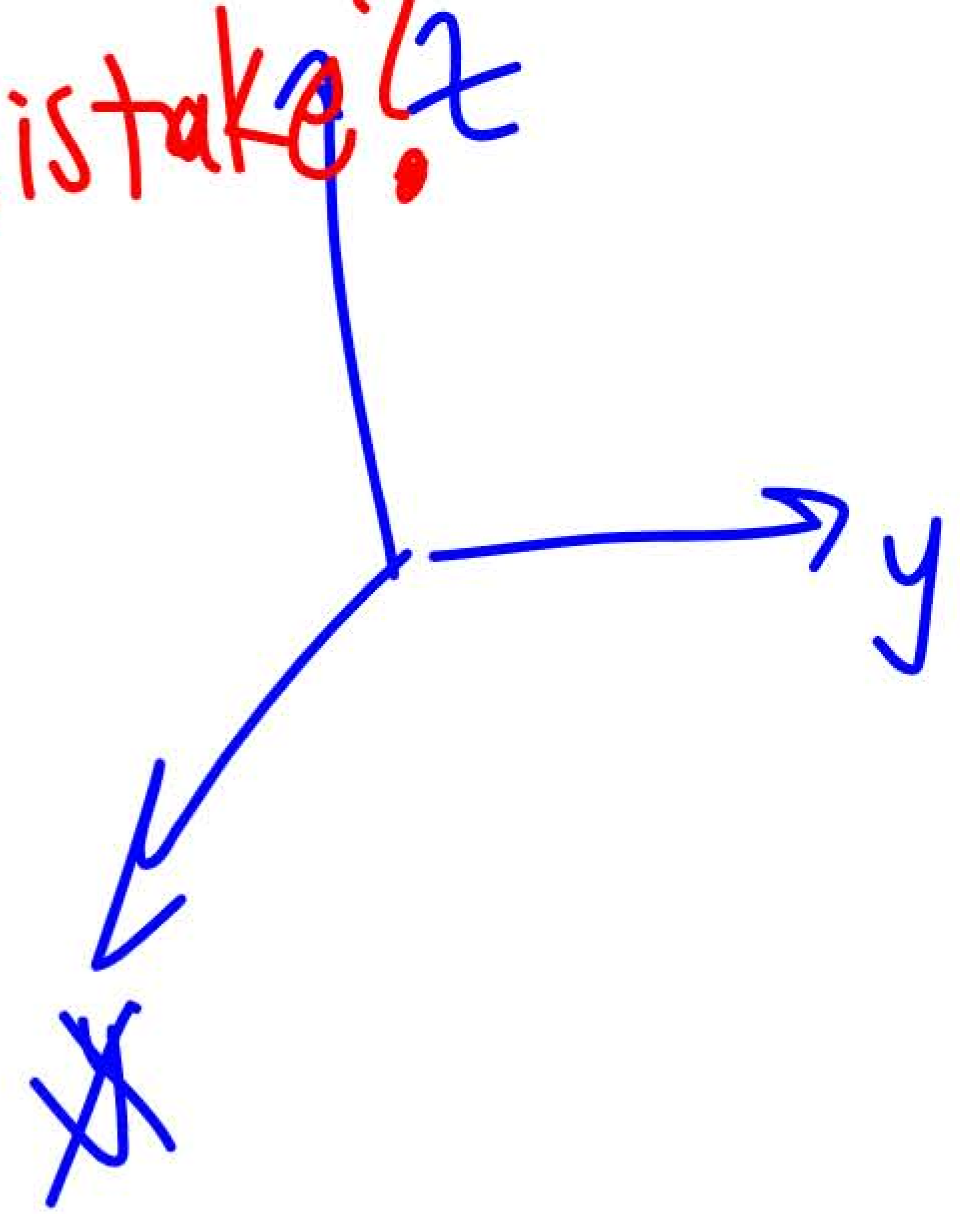
$= \frac{3}{5}(2xy) - \frac{4}{5}x^2$

$D_v f(-5, 5) = -\frac{150}{5} - \frac{100}{5}$

$= -\frac{250}{5}$

$= -50$

Mistake?



D_u

unit vector

4.28 Find the directional derivative $D_{\mathbf{u}} f(x, y)$ of $f(x, y) = 3x^2 y - 4xy^3 + 3y^2 - 4x$ in the direction of $\mathbf{u} = \left(\cos \frac{\pi}{3}\right)\mathbf{i} + \left(\sin \frac{\pi}{3}\right)\mathbf{j}$ using Equation 4.37. What is $D_{\mathbf{u}} f(3, 4)$?

$$= \frac{1}{2}i + \frac{\sqrt{3}}{2}j$$

$$D_u f = \langle 6xy - 4y^3 - 4, 3x^2 - 12xy^2 + 6y \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$D_u f(3,4) = \langle 72 - 256 - 4, 27 - 12 \times 48 + 24 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$= \langle -188, -535 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

$$= -94 - \frac{535\sqrt{3}}{2}$$

$$D_u f = \boxed{\langle f_x, f_y \rangle} \cdot \langle \cos\theta, \sin\theta \rangle$$

Gradient

The right-hand side of **Equation 4.37** is equal to $f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$, which can be written as the dot product of two vectors. Define the first vector as $\boxed{\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}}$ and the second vector as $\mathbf{u} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}$. Then the right-hand side of the equation can be written as the dot product of these two vectors:

$$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.} \tag{4.38}$$

Definition

Let $z = f(x, y)$ be a function of x and y such that f_x and f_y exist. The vector $\nabla f(x, y)$ is called the **gradient** of f and is defined as

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}. \tag{4.39}$$

The vector $\nabla f(x, y)$ is also written as "grad f."

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$

280. Find the gradient of $f(x, y) = \dfrac{14 - x^2 - y^2}{3}$. Then, find the gradient at point $P(1, 2)$.

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \left\langle -\frac{2x}{3}, -\frac{2y}{3} \right\rangle$$

$$\nabla f(1,2) = \left\langle -\frac{2}{3}, -\frac{4}{3} \right\rangle$$

gradient at $P(1,2)$

4.29 Find the gradient $\nabla f(x, y)$ of $f(x, y) = (x^2 - 3y^2)/(2x + y)$.

$$\nabla f = \left\langle \frac{2x(2x+y) - (x^2-3y^2)\cdot y}{(2x+y)^2}, \frac{-6y(2x+y) - (x^2-3y^2)\,1}{(2x+y)^2} \right\rangle$$

quotient rule

$$\langle f_x, f_y \rangle$$

Theorem 4.13: Properties of the Gradient

Suppose the function $z = f(x, y)$ is differentiable at (x_0, y_0) (**Figure 4.41**).

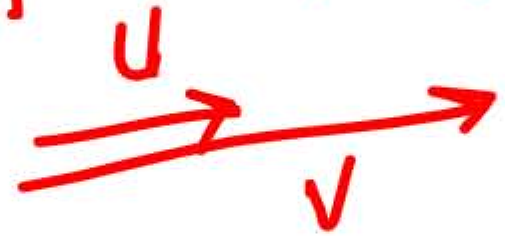
i. If $\nabla f(x_0, y_0) = \mathbf{0}$, then $D_{\mathbf{u}} f(x_0, y_0) = 0$ for any unit vector $\mathbf{u}$.

ii. If $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $D_{\mathbf{u}} f(x_0, y_0)$ is maximized when $\mathbf{u}$ points in the same direction as $\nabla f(x_0, y_0)$. The maximum value of $D_{\mathbf{u}} f(x_0, y_0)$ is $\| \nabla f(x_0, y_0) \|$. $\|u\| = 1$

iii. If $\nabla f(x_0, y_0) \neq \mathbf{0}$, then $D_{\mathbf{u}} f(x_0, y_0)$ is minimized when $\mathbf{u}$ points in the opposite direction from $\nabla f(x_0, y_0)$. The minimum value of $D_{\mathbf{u}} f(x_0, y_0)$ is $-\| \nabla f(x_0, y_0) \|$. $\theta = 180°$

i) $D_u f = \nabla \cdot u = \langle 0, 0 \rangle \cdot u = 0$

$\|u \cdot v\| = \|u\|\|v\| \cdot \cos\theta$ max $\cos\theta = 1$, $\theta = 0°$



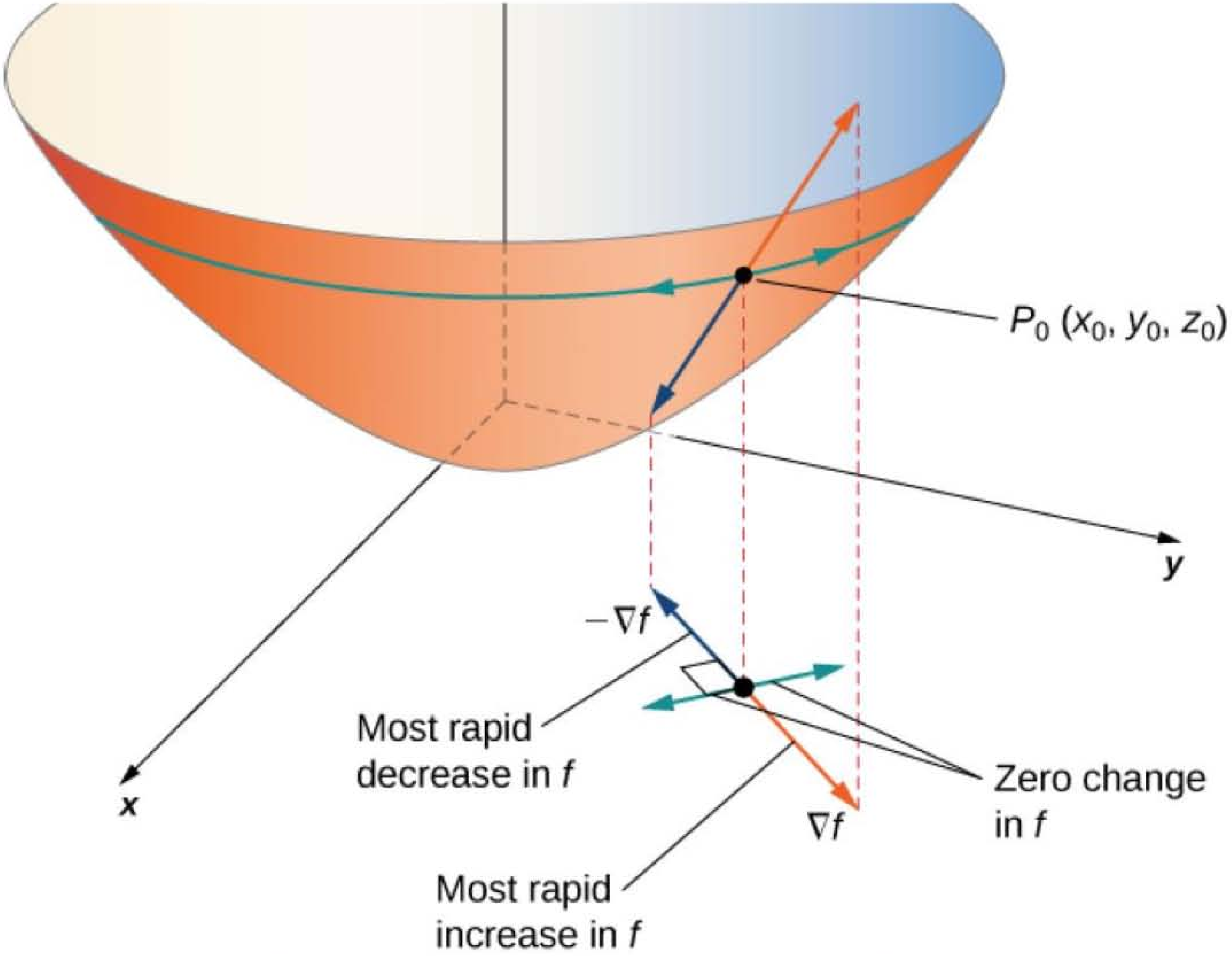


Figure 4.41 The gradient indicates the maximum and minimum values of the directional derivative at a point.

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4.30 Find the direction for which the directional derivative of $g(x, y) = 4x - xy + 2y^2$ at $(-2, 3)$ is a maximum. What is the maximum value?

$$\nabla g = \langle g_x, g_y \rangle = \langle 4 - y, -x + 4y \rangle$$

$$\nabla g(-2,3) = \langle 1, 14 \rangle \qquad u = \frac{\nabla}{\|\nabla\|} = \frac{\langle 1, 14 \rangle}{\sqrt{197}} \quad \text{unit vector}$$

$$\text{maximum value of } D_u g = \|\nabla\| = \sqrt{197}$$

Gradients and Level Curves

$z = \text{constant}$

Define $g(t) = f(x(t),\ y(t))$ and calculate $g'(t)$ on the level curve. By the chain Rule,

$$g'(t) = f_x(x(t),\ y(t))x'(t) + f_y(x(t),\ y(t))y'(t).$$

$$\underbrace{\nabla f(x,\ y)}_{\text{normal}} \cdot \underbrace{\langle\, x'(t),\ y'(t)\,\rangle}_{\text{tangent}} = 0.$$

Theorem 4.14: Gradient Is Normal to the Level Curve

Suppose the function $z = f(x,\ y)$ has continuous first-order partial derivatives in an open disk centered at a point $(x_0,\ y_0)$. If $\nabla f(x_0,\ y_0) \neq \mathbf{0}$, then $\nabla f(x_0,\ y_0)$ is normal to the level curve of f at $(x_0,\ y_0)$.

4.31 For the function $f(x, y) = x^2 - 2xy + 5y^2 + 3x - 2y + 4,$ find the tangent to the level curve at point $(1, 1)$. Draw the graph of the level curve corresponding to $f(x, y) = 8$ and draw $\nabla f(1, 1)$ and a tangent

$z=8$

$\nabla f = \langle 2x-2y+3, -2x+10y-2 \rangle$

$\nabla f(1,1) = \langle 3, 6 \rangle$ normal.

$\langle -6, 3 \rangle$ tangent