5 | MULTIPLE INTEGRATION

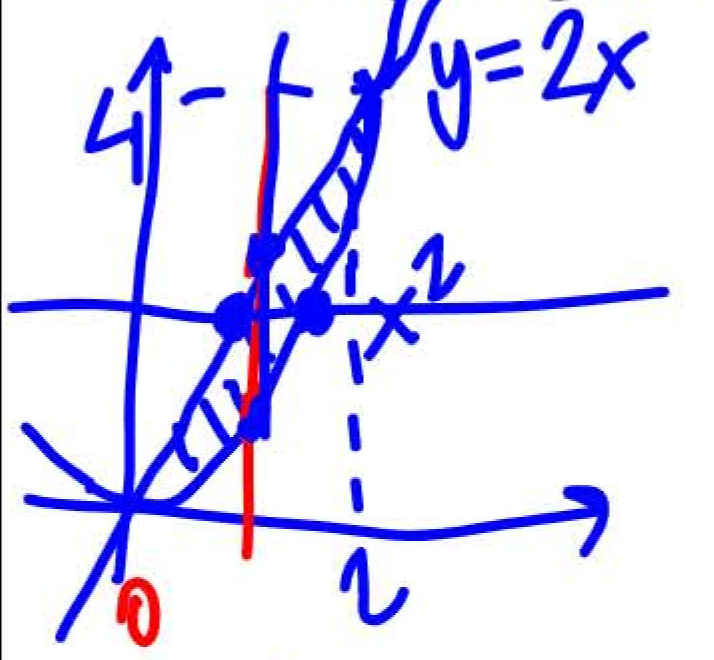
5.1 D. I rectangular
5.2 " " general regions

5.3 | Double Integrals in Polar Coordinates

Slide 1

100%

5.11 Evaluate the iterated integral $\iint_D (x^2+y^2)\,dA$ over the region D in the first quadrant between the functions $y=2x$ and $y=x^2$. Evaluate the iterated integral by integrating first with respect to y and then integrating first with resect to x.



$$\int_0^2\int_{x^2}^{2x}(x^2+y^2)\,dy\,dx$$

$$=\int_0^2\left(x^2y+\frac{y^3}{3}\right)\Big|_{x^2}^{2x}dx=\int_0^2\left(2x^3+\frac{8x^3}{3}-x^4-\frac{x^6}{3}\right)dx$$

$x^2\le y\le 2x$

$0\le x\le 2$

$$=\left(\frac{14}{3}\frac{x^4}{4}-\frac{x^5}{5}-\frac{x^7}{3\times 7}\right)\Big|_0^2=\frac{7}{6}16-\frac{32}{5}-\frac{128}{21}=$$

$\frac{y}{2}\le x\le\sqrt{y}$

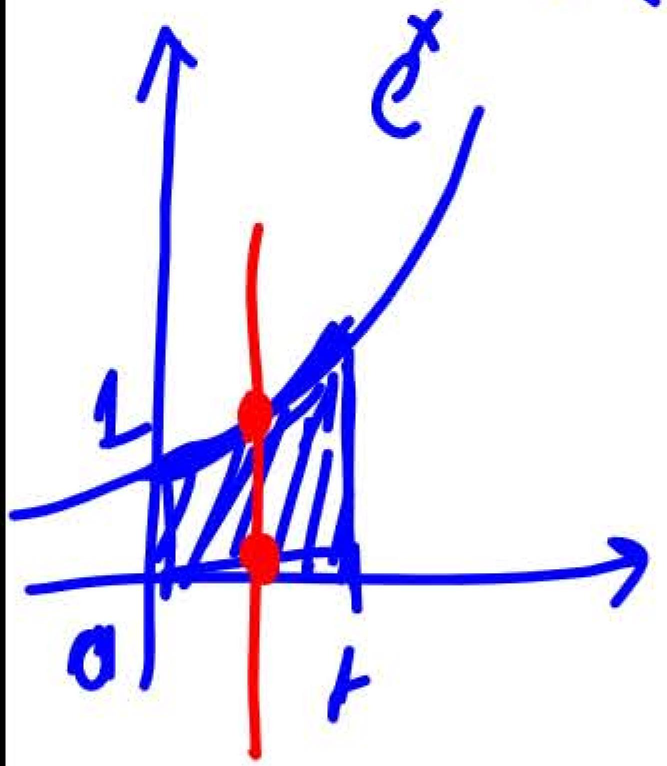
$0\le y\le 4$

$$\int_0^4\int_{y/2}^{\sqrt{y}}(x^2+y^2)\,dx\,dy$$

5.12 Find the volume of the solid bounded above by $f(x, y) = 10 - 2x + y$ over the region enclosed by the curves $y = 0$ and $y = e^x$, where x is in the interval $[0, 1]$.

$0 \leq y \leq e^x$ $\qquad 0 \leq x \leq 1$

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + c$$



$$V = \int_0^1 \int_0^{e^x} (10 - 2x + y)\,dy\,dx$$

$$\int_0^1 \left((10-2x)y + \frac{y^2}{2} \right)\Big|_0^{e^x} dx = \int_0^1 \left[(10-2x)e^x + \frac{e^{2x}}{2} \right) dx$$

$$= (10e^x)\Big|_0^1 - \underbrace{\int_0^1 2xe^x}_{} + \frac{e^{2x}}{4}\Big|_0^1$$

$$= 10e - 10 + \frac{e^2}{4} - \frac{1}{4} - 2$$

$$\int xe^x\,dx = xe^x - \int e^x\,dx$$

integration by parts

$u = x \qquad du = dx$

$dv = e^x\,dx \qquad v = e^x$

$$= (xe^x - e^x)\Big|_0^1$$

$$= (e - e) - (0 - 1)$$

$$= 1$$

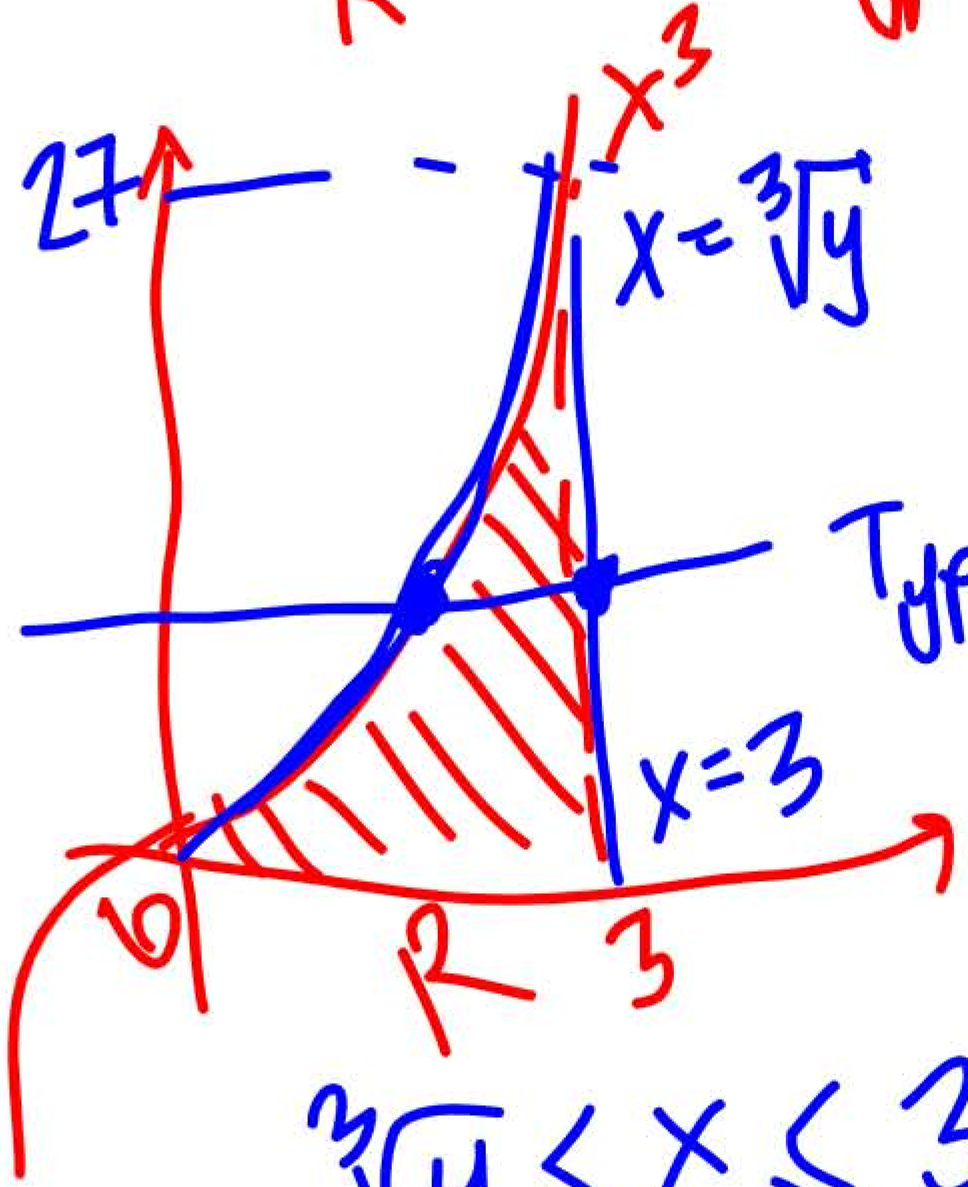
100%

5.13 Find the area of a region bounded above by the curve $y = x^3$ and below by $y = 0$ over the interval $[0, 3]$.

$$A = \iint_R 1\, dA$$

Type I

$$\int_0^3 \int_0^{x^3} 1\, dy\, dx = \int_0^3 y \Big|_0^{x^3} dx = \int_0^3 x^3\, dx = \frac{x^4}{4}\Big|_0^3 = \frac{81}{4}$$



$\sqrt[3]{y} \le x \le 3$

$0 \le y \le 27$

Type II

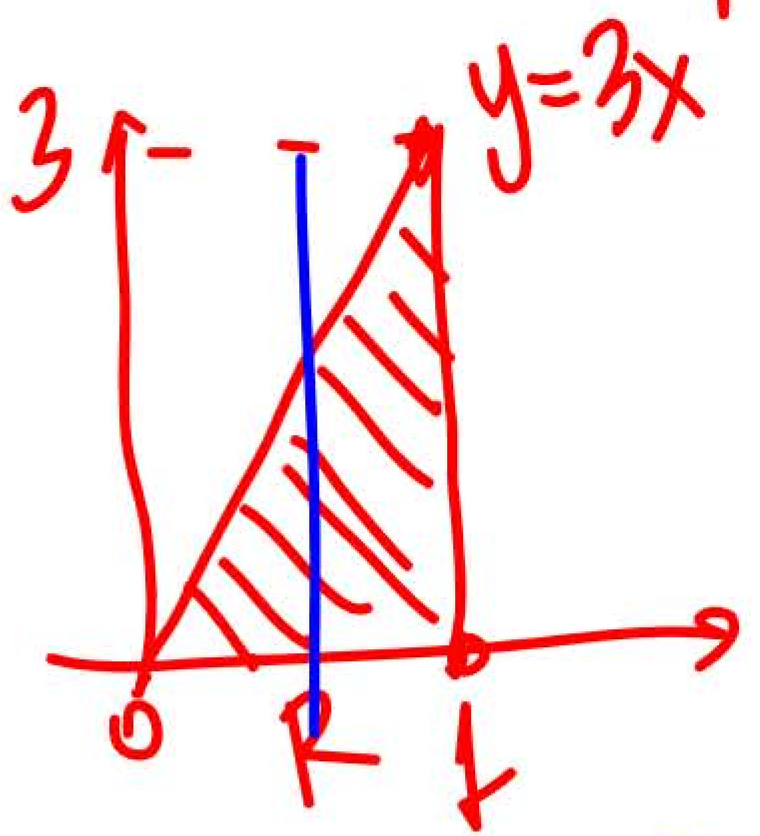
$$\int_0^{27} \int_{\sqrt[3]{y}}^{3} 1.\, dx\, dy = \int_0^{27} x \Big|_{\sqrt[3]{y}}^{3} dy$$

$$= \int_0^{27} (3 - y^{1/3})\, dy = \left(3y - \frac{y^{4/3}}{4/3}\right)\Big|_0^{27}$$

$$= 81 - \frac{3}{4}(3^{\cancel{3}})^{4/\cancel{3}} = \frac{81}{4}$$

5.14 Find the average value of the function $f(x, y) = xy$ over the triangle with vertices (0, 0), (1, 0) and (1, 3).

$$f_{ave} = \frac{1}{\text{Arear of } R}\iint_R f(x,y)\,dA = \frac{1}{\frac{3}{2}}\int_0^1\int_0^{3x} xy\,dy\,dx$$



$$= \frac{2}{3}\int_0^1 \left(\frac{xy^2}{2}\Big|_0^{3x}\right)dx = \frac{2}{3}\int_0^1 \frac{9x^3}{2}\,dx$$

$$= 3\,\frac{x^4}{4}\Big|_0^1 = 3\left(\frac{1}{4}\right)$$

is the average value of f over R.

$0 \le y \le 3x$
$0 \le x \le 1$

$\frac{y}{3} \le x \le 1$
$0 \le y \le 3$ Type II

Improper Double Integrals

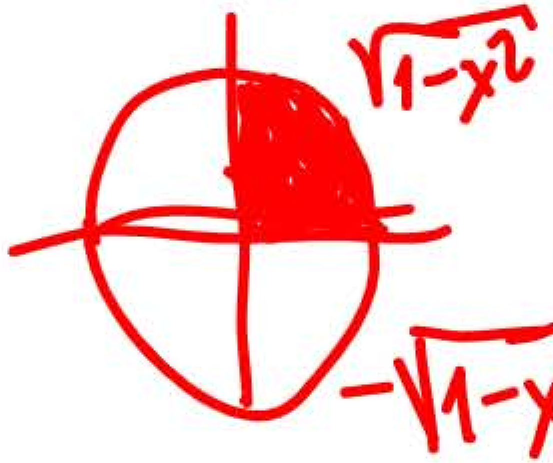
Theorem 5.7: Improper Integrals on an Unbounded Region

If R is an unbounded rectangle such as $R = \{(x, y): a \le x \le \infty,\ c \le y \le \infty\}$, then when the limit exists, we have

$$\iint\limits_R f(x, y)\,dA = \lim_{(b,\,d) \to (\infty,\,\infty)} \int_a^b \left(\int_c^d f(x, y)\,dy \right) dx = \lim_{(b,\,d) \to (\infty,\,\infty)} \int_c^d \left(\int_a^b f(x, y)\,dy \right) dy.$$

5.15 Evaluate the improper integral $\iint\limits_D \frac{y}{\sqrt{1 - x^2 - y^2}}\,dA$ where $D = \left\{(x, y) x \ge 0,\ y \ge 0,\ x^2 + y^2 \le 1\right\}$.

when $x^2+y^2=1$, f is undefined



$$\int_0^1 \int_{0}^{\sqrt{1-x^2}} \frac{y}{\sqrt{1-x^2-y^2}}\,dy\,dx$$

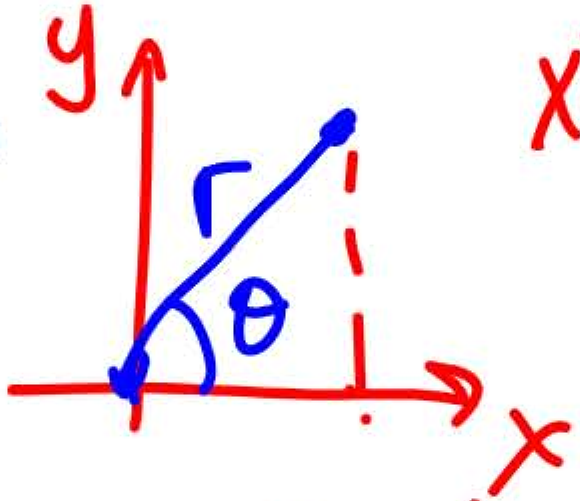
$$\int_0^1 -\frac{\sqrt{1-x^2-y^2}}{2}\Bigg|_{-x^2}^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 0\,dx = 0$$

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \cancel{\frac{\partial x}{\partial \theta}} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

Jacobian

$x = r\cos\theta$ $y = r\sin\theta$



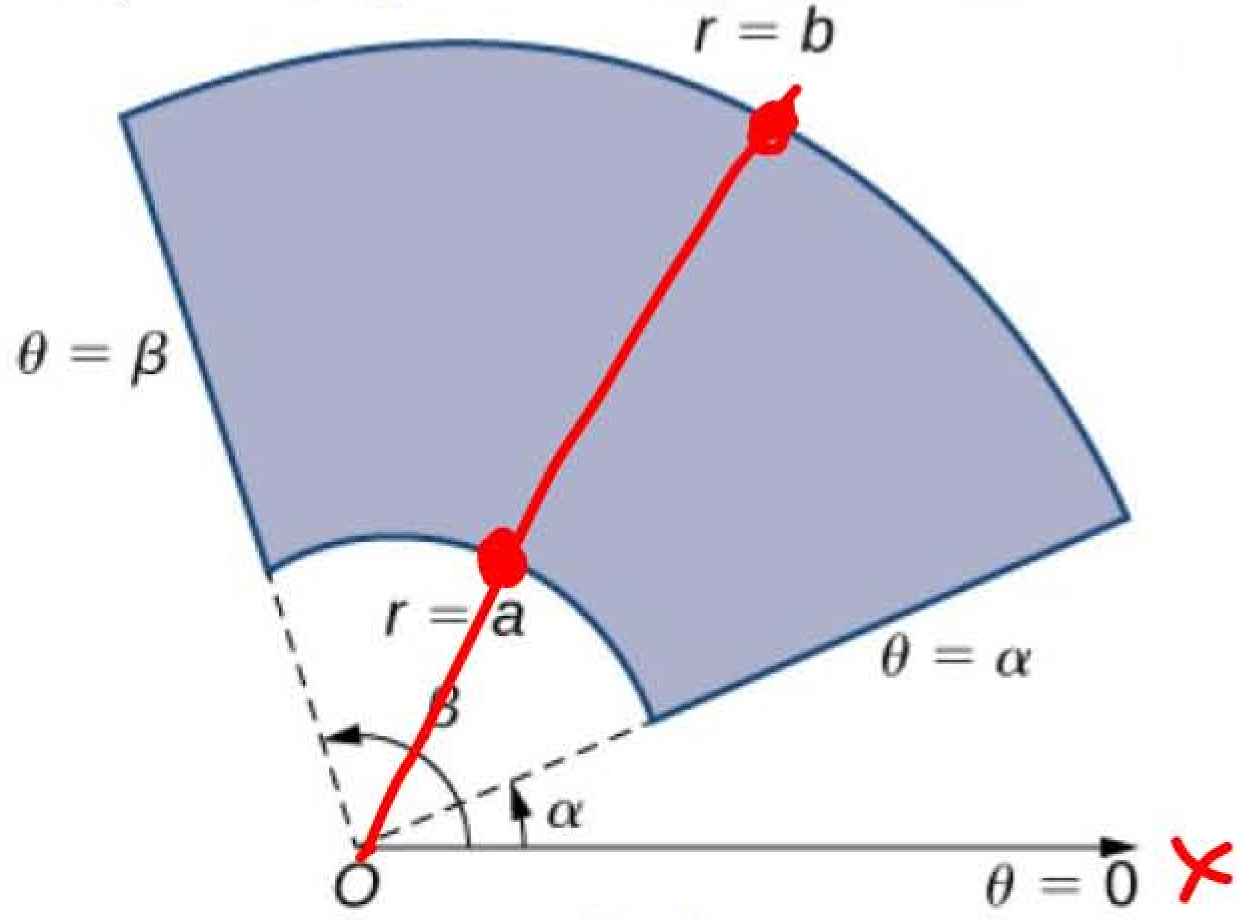
$x^2 + y^2 = r^2$

$\tan\theta = \dfrac{y}{x}$

using $x = r\cos\theta$, $y = r\sin\theta$, and $dA = r\,dr\,d\theta$ changes it to

$$\iint_R f(x, y)dA = \iint_R f(r\cos\theta,\ r\sin\theta)r\,dr\,d\theta.$$

$\alpha \le \theta \le \beta$ $a \le r \le b$



rectangular region

$$\begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

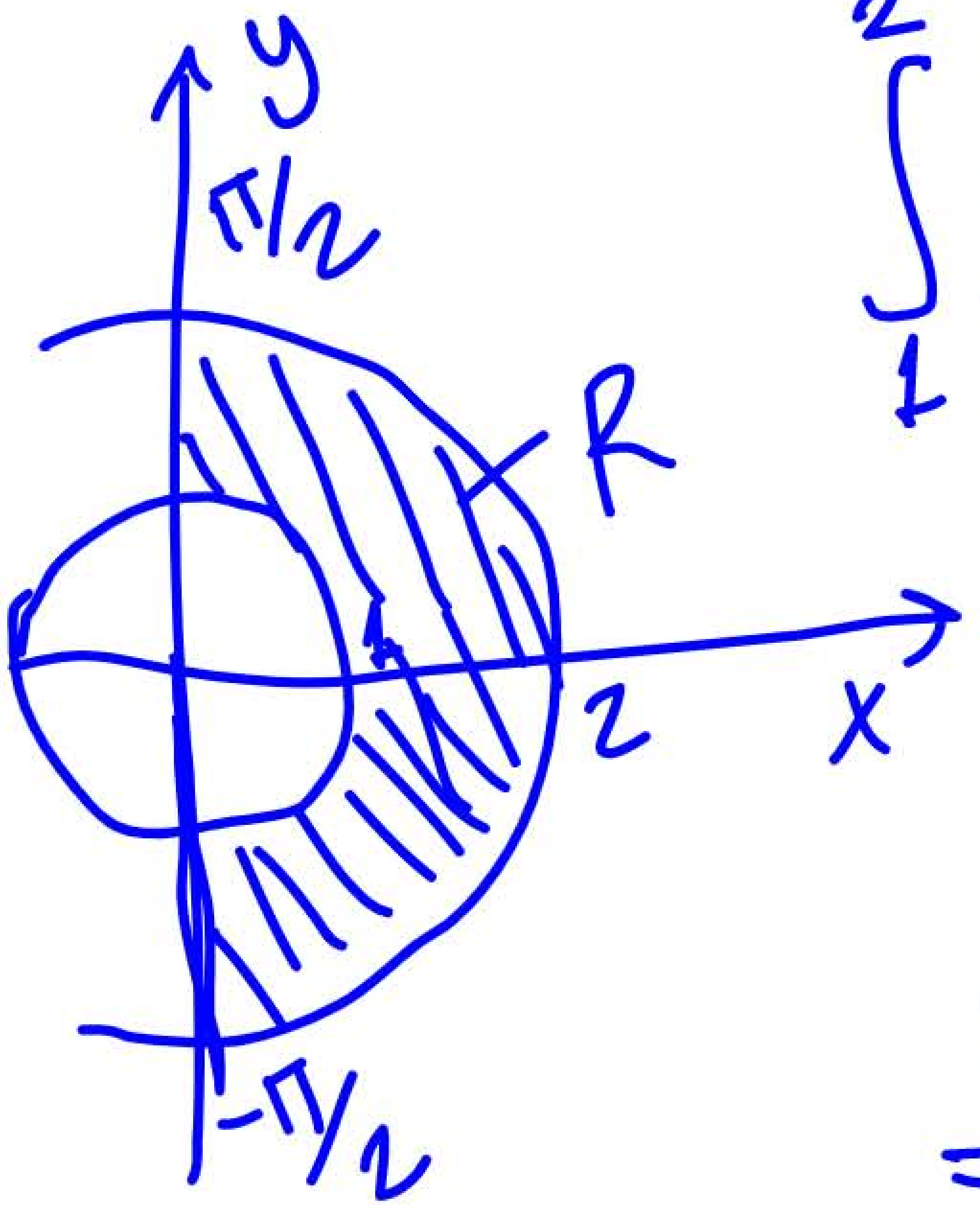
$$dy\,dx = dx\,dy = \boxed{r\,dr\,d\theta = dA}$$

$$\iint_R f(r, \theta)dA = \iint_R f(r, \theta)r\,dr\,d\theta = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=a}^{r=b} f(r, \theta)r\,dr\,d\theta.$$

Slide 8

5.17 Sketch the region $R = \left\{(r, \theta) \middle| 1 \le r \le 2,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$, and evaluate $\iint_R \underbrace{x\,dA}$.

$r\,dr\,d\theta$



$$\int_1^2 \int_{-\pi/2}^{\pi/2} r\cos\theta\, r\, d\theta\, dr$$

$$= \int_1^2 r^2 \sin\theta \Big|_{-\pi/2}^{\pi/2} dr$$

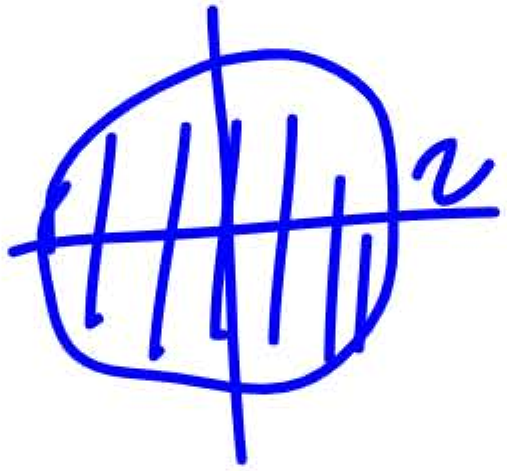
$$= \int_1^2 r^2 (1-(-1))\,dr = \frac{2r^3}{3}\Big|_1^2 = \frac{16-2}{3} = \frac{14}{3}$$

a

5.18 Evaluate the integral $\iint_R \left(4-x^2-y^2\right)dA$ where R is the circle of radius 2 on the xy-plane.

$0 \le r \le 2$

$0 \le \theta \le 2\pi$



$$\iint_R \left(4-(x^2+y^2)\right)dA = \int_0^{2\pi}\int_0^{2}(4-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2}(4r-r^3)\,dr\,d\theta = \int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\Big|_0^{2}\,d\theta$$

$$= \int_0^{2\pi}\left[8-4-0\right]d\theta = 4\theta\Big|_0^{2\pi} = 8\pi$$

General Polar Regions of Integration

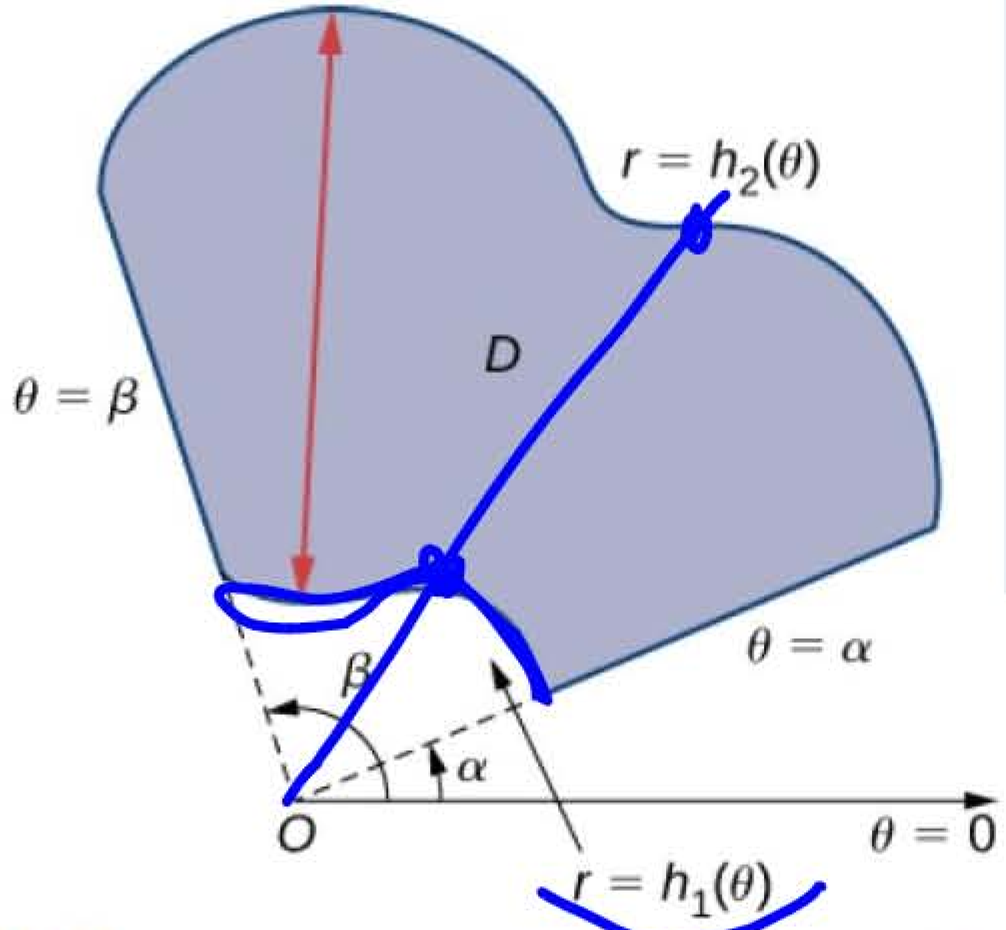


Figure 5.32 A general polar region between $\alpha < \theta < \beta$ and $h_1(\theta) < r < h_2(\theta)$.

Theorem 5.8: Double Integrals over General Polar Regions

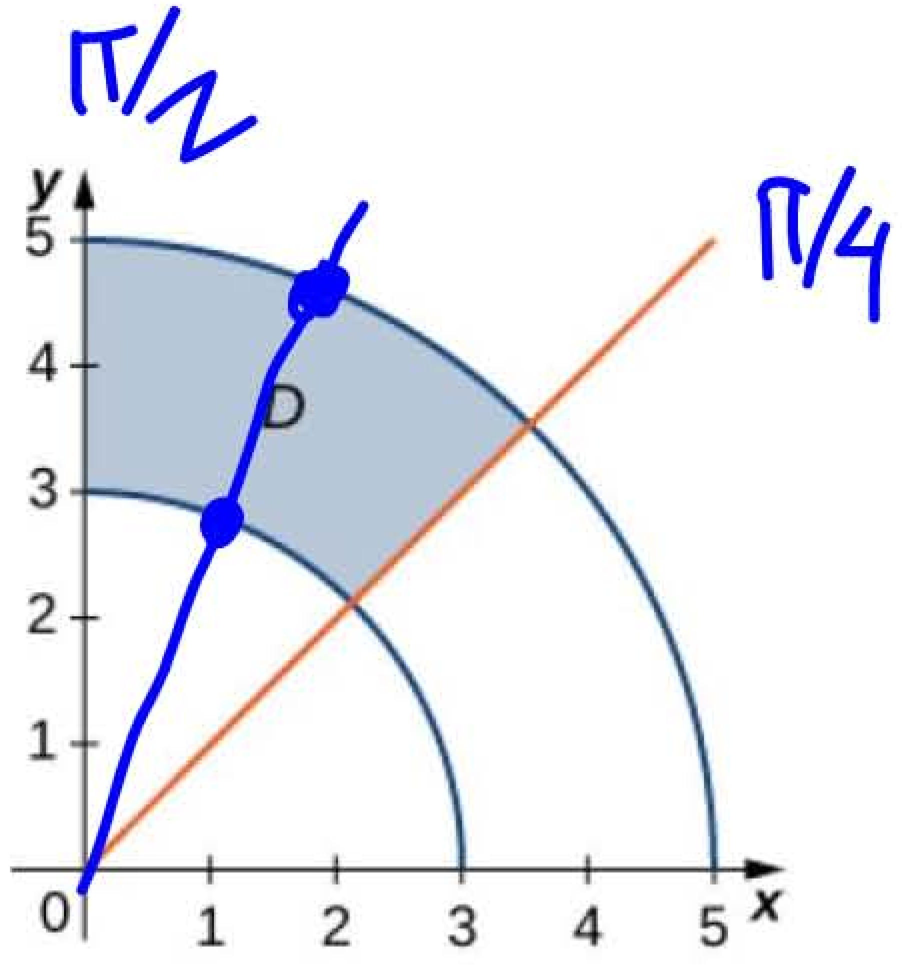
If $f(r, \theta)$ is continuous on a general polar region D as described above, then

$$\iint_D f(r, \theta) r \, dr \, d\theta = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} f(r, \theta) r \, dr \, d\theta$$

$$\int_0^{\pi} \int_{\theta}^{\cos\theta} \cdots \; r\,dr\,d\theta$$

In the following exercises, the graph of the polar rectangular region D is given. Express D in polar coordinates.

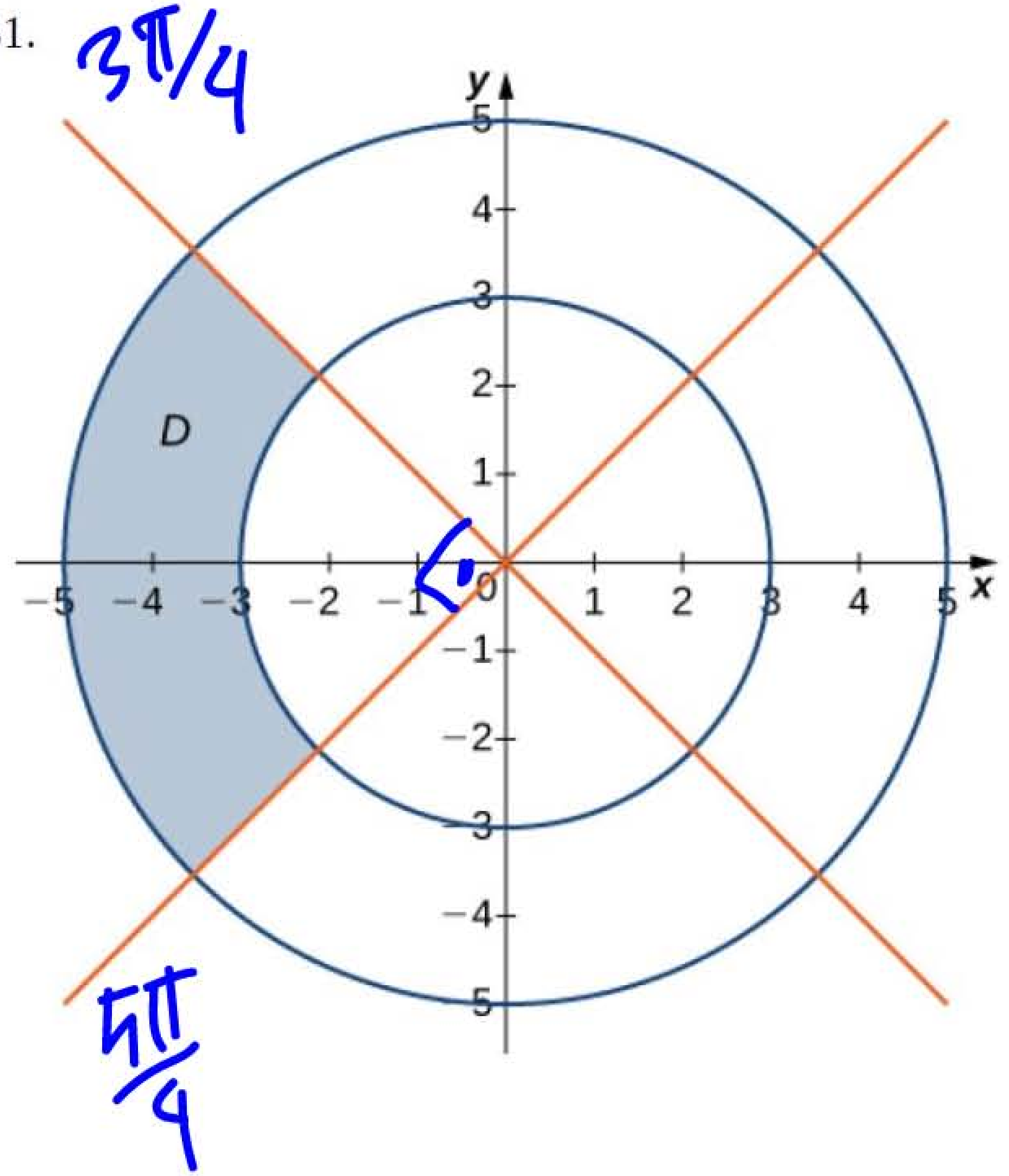
129.



$\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$

$3 \leq r \leq 5$

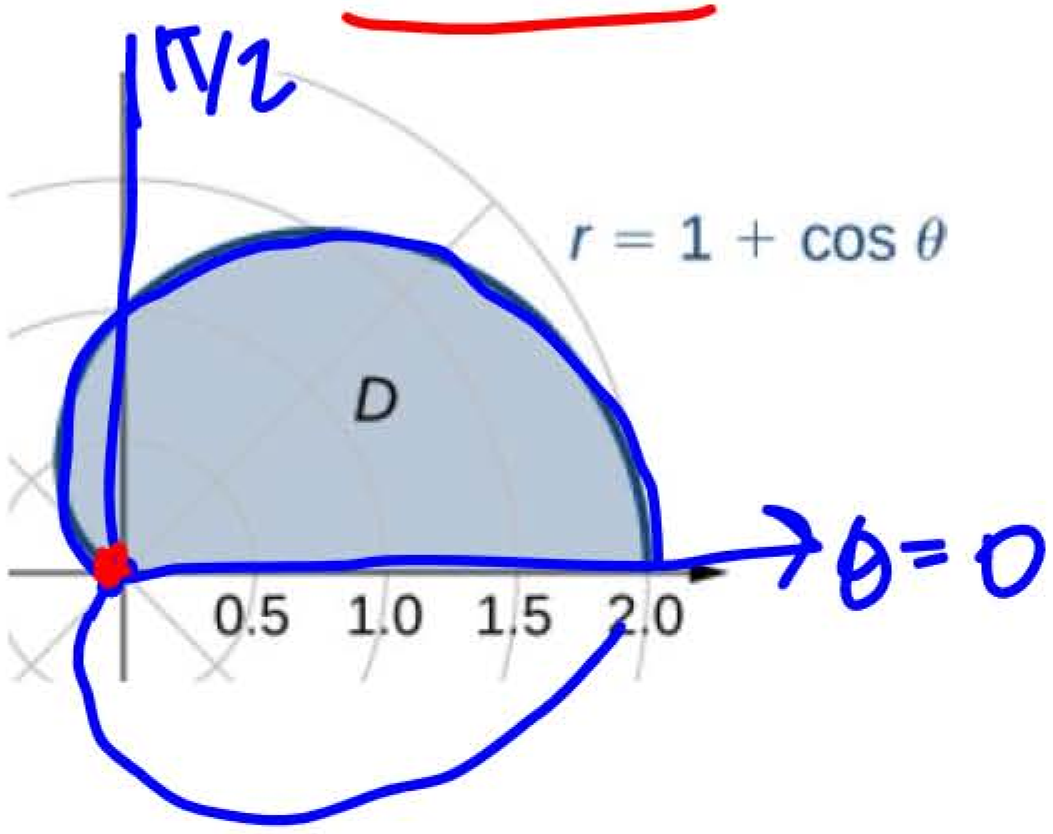
131.



$3 \leq r \leq 5$

$\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$

Evaluate the integral $\iint_D r^2 \sin\theta\, r\, dr\, d\theta$ where D is the region bounded by the polar axis and the upper half of the cardioid $r = 1 + \cos\theta$.



$0 \leq \theta \leq \pi$

$0 \leq r \leq 1+\cos\theta$

$$\int_0^{\pi}\int_0^{1+\cos\theta} r^3 \sin\theta\, dr\, d\theta$$

$$= \int_0^{\pi} \frac{r^4}{4}\sin\theta \Big|_0^{1+\cos\theta} d\theta = \int_0^{\pi} \frac{(1+\cos\theta)^4}{4} \sin\theta\, d\theta$$

$u = 1+\cos\theta$

$du = -\sin\theta\, d\theta$

$\theta = 0 \Rightarrow u = 1+1 = 2$

$\theta = \pi \Rightarrow u = 1-1 = 0$

$$= \int_2^0 \frac{u^4}{4}(-du) = \int_0^2 \frac{u^4}{4} du$$

$$= \frac{u^5}{20}\Big|_0^2 = \frac{32}{20} = \frac{8}{5}$$

a1468.roc2.blindsidenetworks.com is now full screen (Press Esc to exit full screen mode)

5.19 Evaluate the integral

$$\iint_D r^2 \sin^2 2\theta r \, dr \, d\theta \text{ where } D = \{(r, \theta) \mid 0 \le \theta \le \pi,\ 0 \le r \le 2\sqrt{\cos 2\theta}\}.$$

$$\int_0^{\pi}\int_0^{2\sqrt{\cos 2\theta}} r^3 \sin^2 2\theta \, dr \, d\theta = \int_0^{\pi} \frac{r^4}{4} \sin^2 2\theta \Big|_0^{2\sqrt{\cos 2\theta}} d\theta$$

$$2\cos 2\theta \sin 2\theta = \sin 4\theta$$

$$= \int_0^{\pi} \frac{16 \cos^2 2\theta \sin^2 2\theta}{4} d\theta = \int_0^{\pi} 2 \sin^2 4\theta \, d\theta$$

trigonometric int.

$$\cos 2A = 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\int_0^{\pi} 2 \, \frac{1 - \cos 8\theta}{2} \, d\theta = \left(\theta - \frac{\sin 8\theta}{8}\right)\Big|_0^{\pi}$$

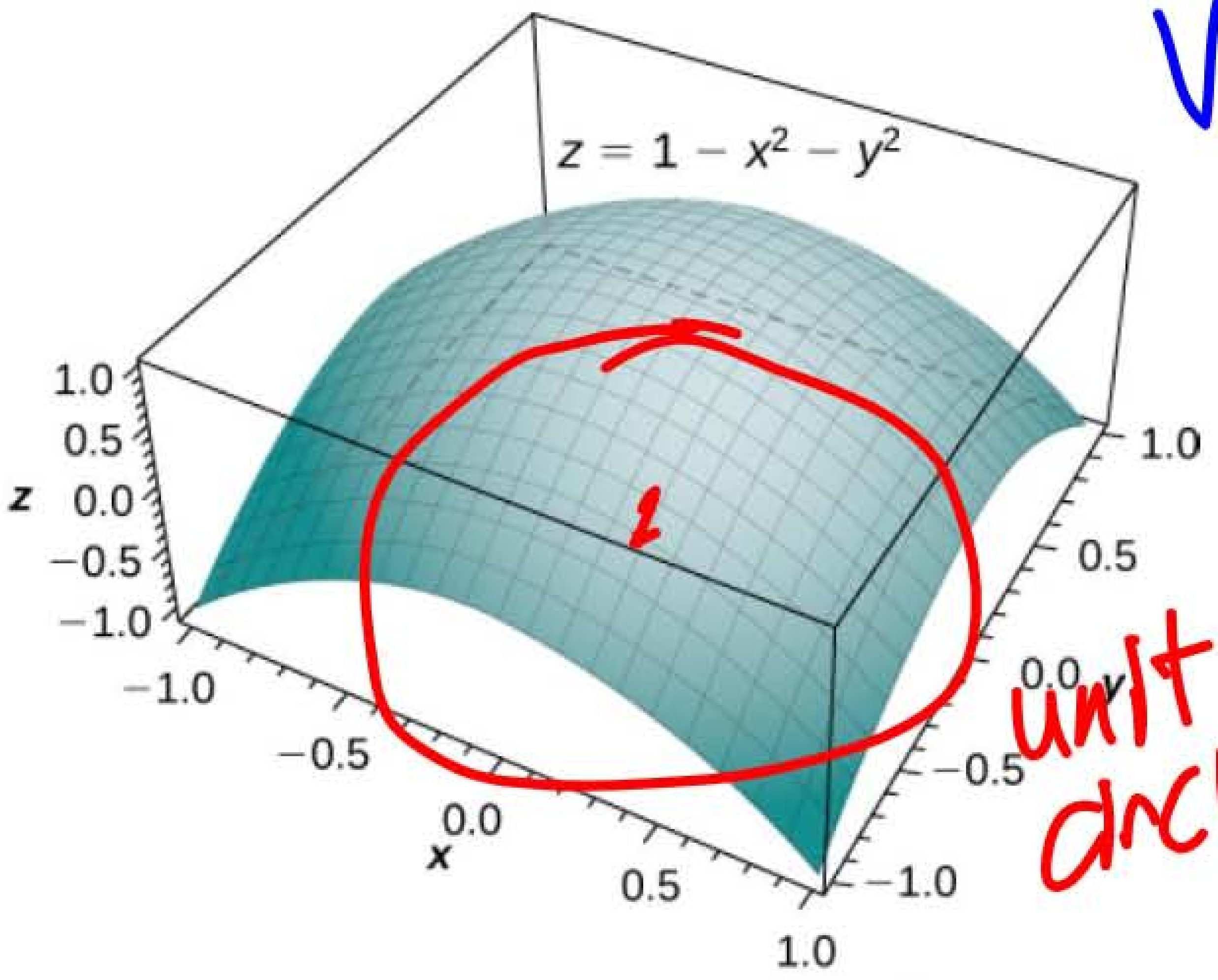
$$\sin 8\pi = 0 = \sin 0$$

$$= \pi$$

Slide 14

100%

Find the volume of the solid that lies under the paraboloid $z = 1 - x^2 - y^2$ and above the unit circle on the xy-plane (see the following figure).



unit circle R

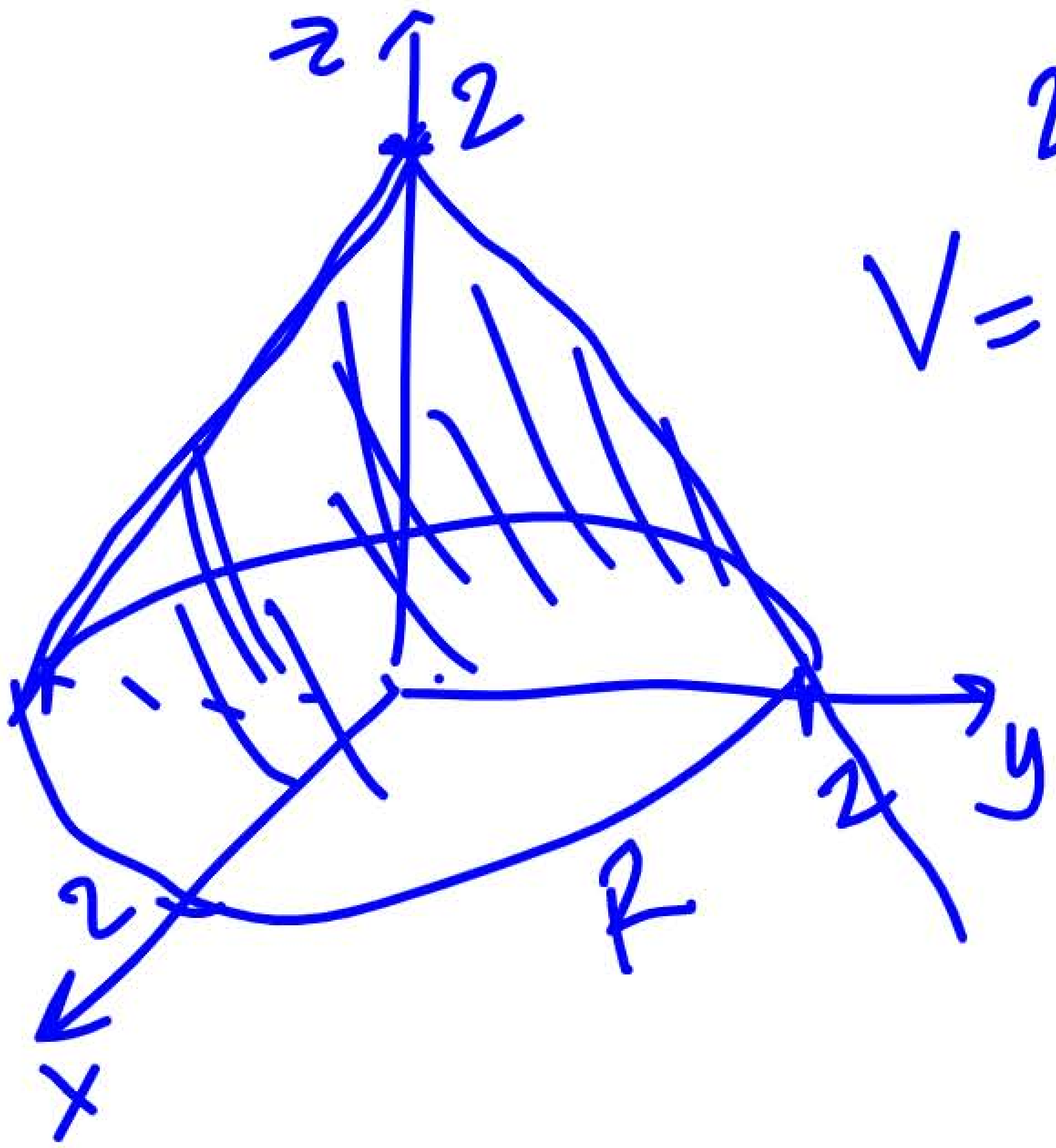
Figure 5.34 The paraboloid $z = 1 - x^2 - y^2$.

$$V = \iint_R (1-(x^2+y^2))\,dA = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 (r-r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Bigg|_0^1 d\theta$$

$$\int_0^{2\pi}\frac{1}{4}\,d\theta = \frac{\theta}{4}\Bigg|_0^{2\pi} = \frac{\pi}{2}\ \text{unit}^3.$$

Use polar coordinates to find the volume inside the cone $z = 2 - \sqrt{x^2 + y^2}$ and above the xy-plane.



$$V = \int_0^{2\pi}\int_0^{2} \left(2 - \sqrt{r^2}\right) r\,dr\,d\theta$$

$$(2r - r^2)\,dr\,d\theta$$

$$\int_0^{2\pi} \left(r^2 - \frac{r^3}{3}\right)\Big|_0^2 \,d\theta$$

$$\int_0^{2\pi} \left(4 - \frac{8}{3}\right) d\theta = \frac{4}{3}\theta \Big|_0^{2\pi} = \frac{8\pi}{3} \text{ unit}^3.$$

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy-plane. See the paraboloid in **Figure 5.35** intersecting the cylinder $(x-1)^2 + y^2 = 1$ above the xy-plane.

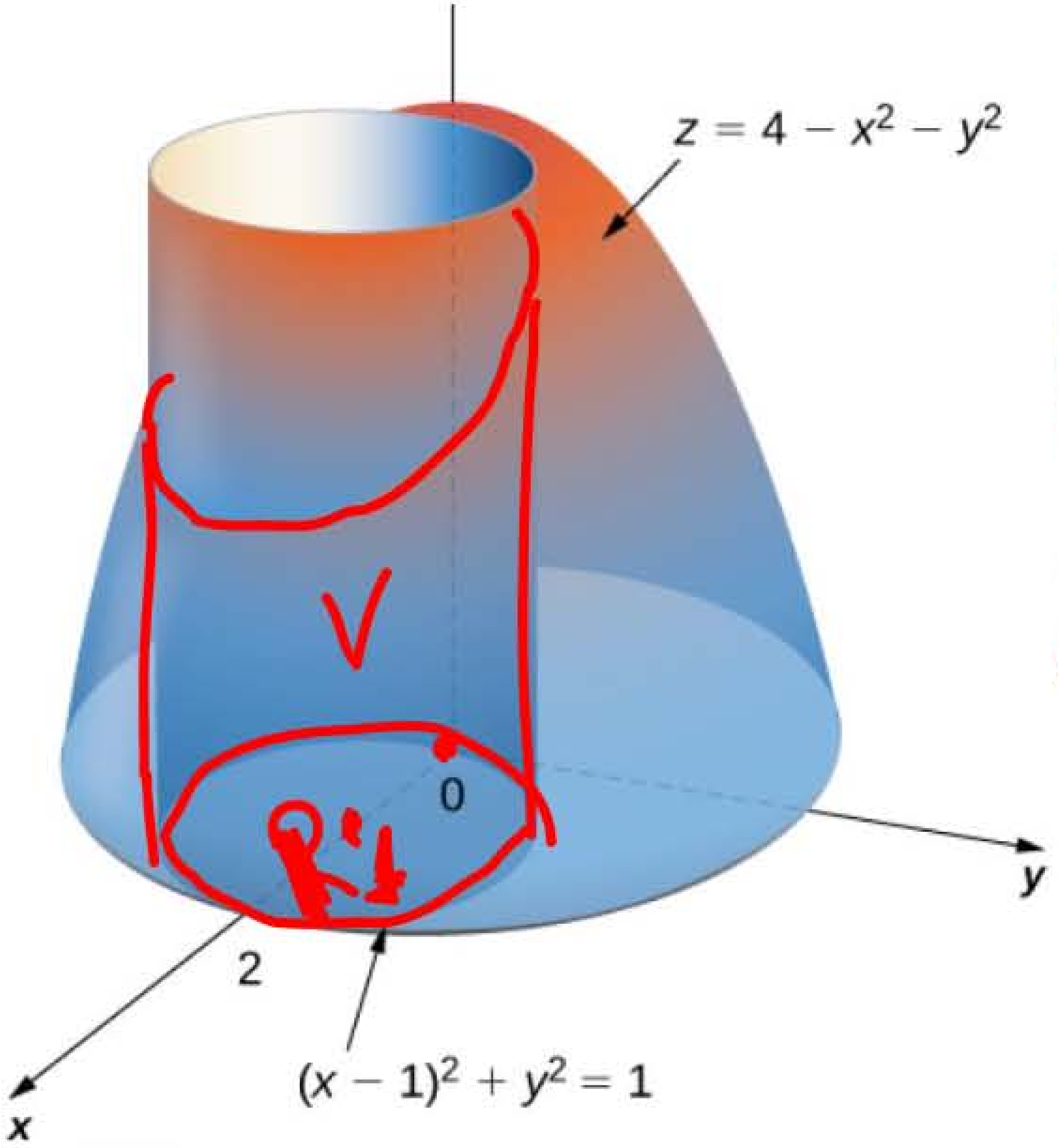


Figure 5.35 Finding the volume of a solid with a paraboloid cap and a circular base.

$$V = \iint_R \left(4 - (x^2 + y^2)\right) dA$$

$$(x-1)^2 + y^2 = 1 \quad R$$

$$x^2 - 2x + \cancel{1} + y^2 = \cancel{1}$$

$$x^2 + y^2 = 2x$$

$$r^2 = 2r\cos\theta$$

$$r = 0 \qquad r = 2\cos\theta$$

$$0 \le r \le 2\cos\theta$$

$$0 \le \theta \le 2\pi$$

$$V=\int_0^{2\pi}\int_0^{2\cos\theta}(4-r^2)\,r\,dr\,d\theta=\int_0^{2\pi}\int_0^{2\cos\theta}(4r-r^3)\,dr\,d\theta$$

$$\cos 2A=2\cos^2 A-1$$

$$\frac{\cos 2A+1}{2}=\cos^2 A$$

$$\int_0^{2\pi}\left(2r^2-\frac{r^4}{4}\right)\Big|_0^{2\cos\theta}d\theta=\int_0^{2\pi}\left(8\cos^2\theta-\frac{16\cos^4\theta}{4}\right)d\theta$$

$$=\int_0^{2\pi}\left(4(\cos 2\theta+1)-\cancel{4}\left(\frac{\cos 2\theta+1}{\cancel{2}}\right)^2\right)d\theta$$

$$=4\left(\frac{\sin 2\theta}{2}+\theta\right)\Big|_0^{2\pi}-\int_0^{2\pi}(\cos^2 2\theta+2\cos 2\theta+1)\,d\theta$$

$$=8\pi-\left[\frac{\sin 4\theta+\theta}{2}+\sin 2\theta+\theta\right]\Big|_0^{2\pi}=8\pi-3\pi=5\pi\ \text{unit}^3$$