5 | MULTIPLE INTEGRATION

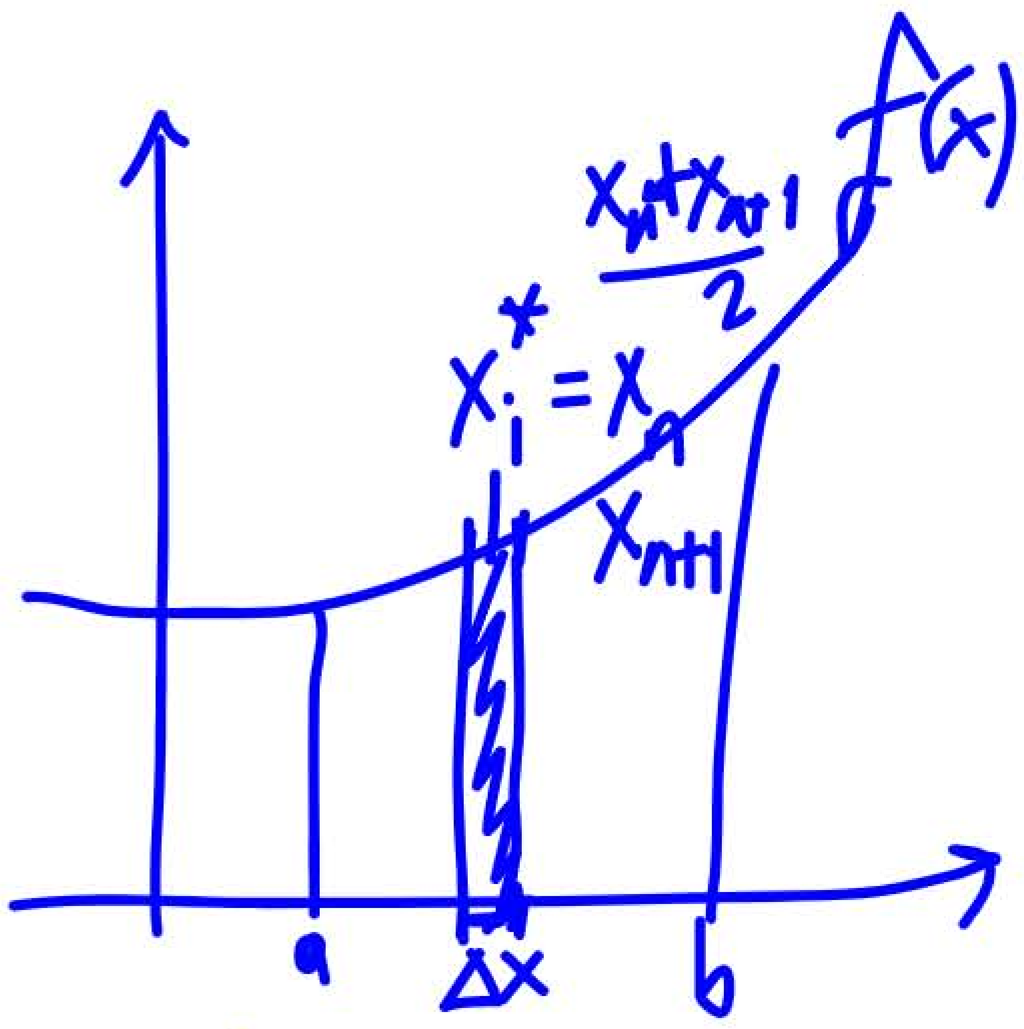
Chapter Outline

$$\int_0^1 \int_1^2 \ldots \, dx\,dy \qquad \iiint_R \ldots \, dx\,dy\,dz$$

Slide 1

100%

Calculus 2



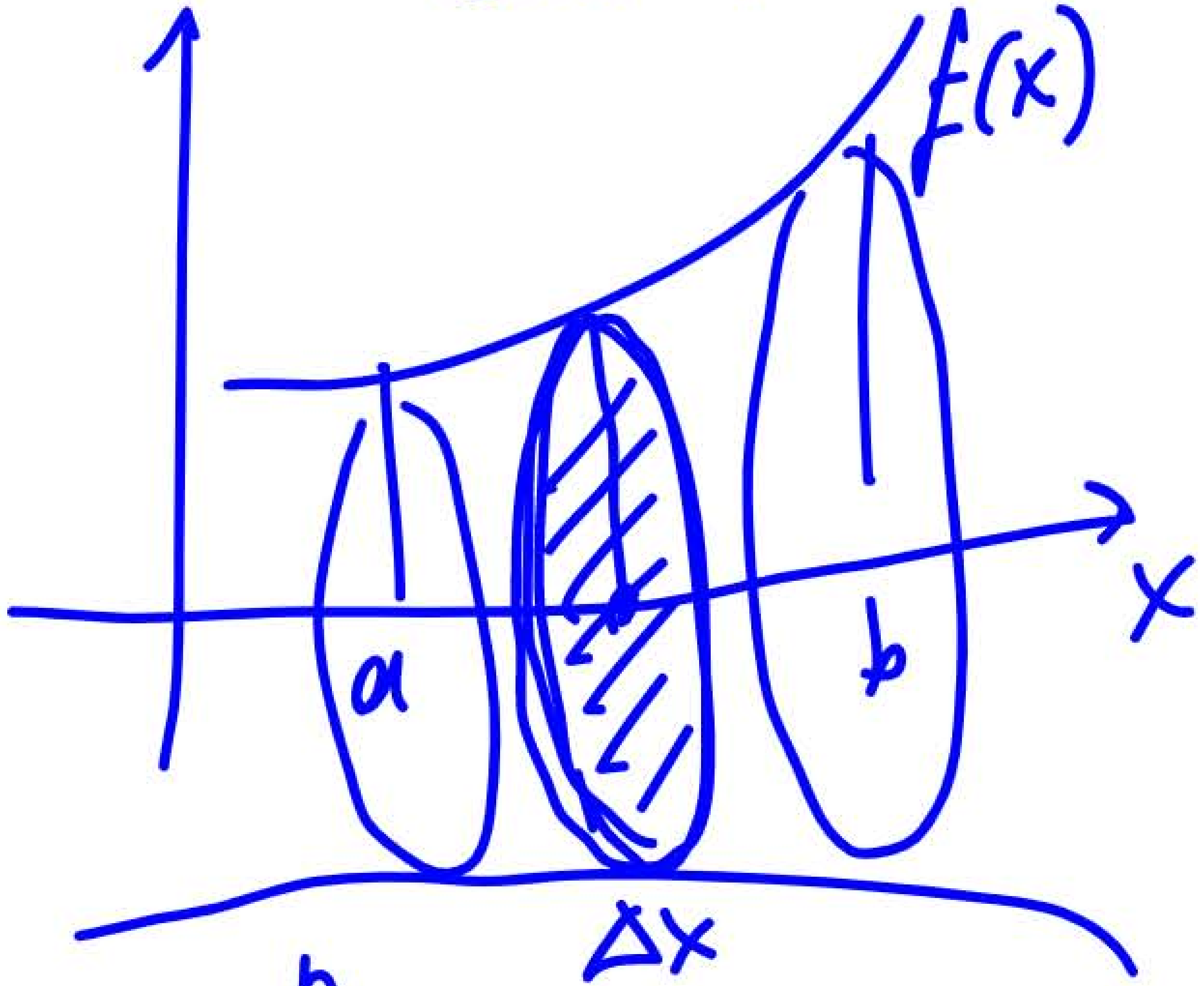
area $\sum_{i=1}^{n} f(x).\Delta x$

Rieman Sum

$$A = \int_a^b f(x)dx$$

Volum

disc method



$$V = \sum_{i=1}^{n} \pi f^2(x) \Delta x$$

$$V = \int_a^b \pi f^2(x) dx$$

Volumes and Double Integrals

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$$

$\Delta A = \Delta x \Delta y$

$dA = dx\,dy$

$\Delta V = \underbrace{f(x,y)}_{\text{height}}\underbrace{\Delta A}_{\text{base area}}$

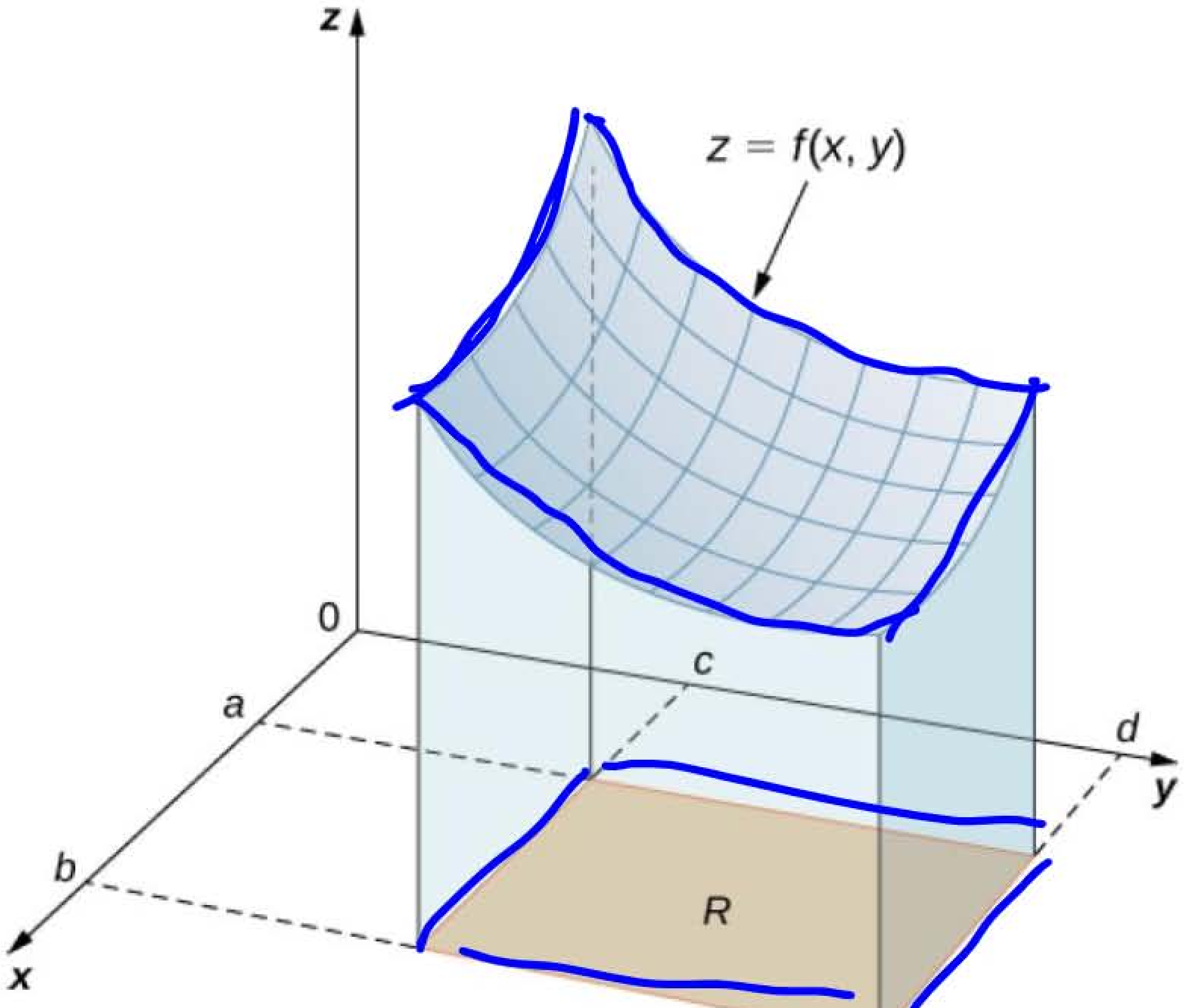
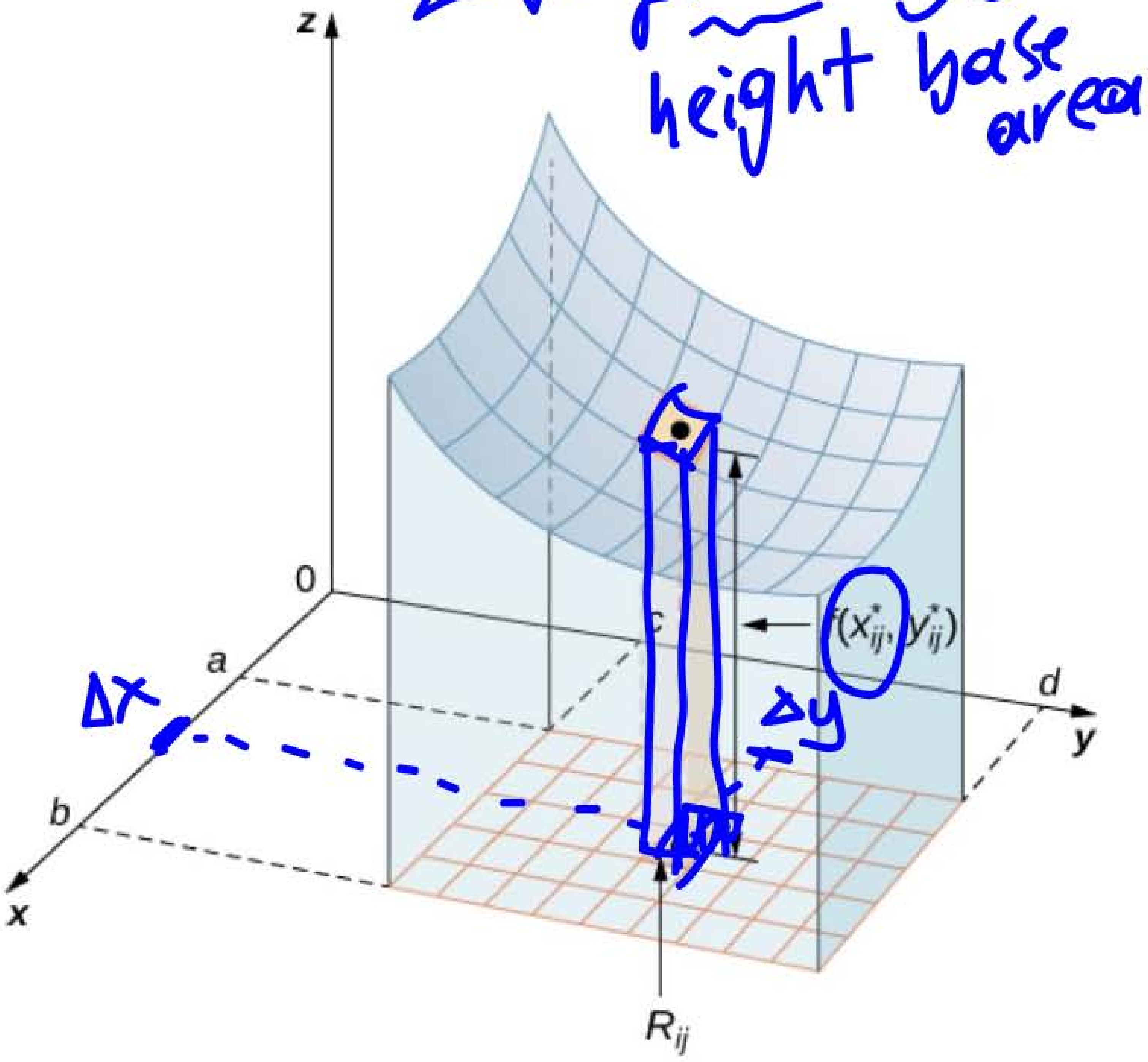


Figure 5.2 The graph of $f(x, y)$ over the rectangle R in the xy-plane is a curved surface.



ΔV

$$V = \lim_{m,\, n \to \infty} \underbrace{\sum_{i=1}^{m}}_{\text{sigma}} \sum_{j=1}^{n} \underline{f(x_{ij}^*,\, y_{ij}^*)\Delta A} \text{ or } V = \lim_{\Delta x,\, \Delta y \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*,\, y_{ij}^*)\Delta A$$

Riemann sum

$$V = \iint_R f(x,y)\, dA$$

Definition

The **double integral** of the function $f(x, y)$ over the rectangular region R in the xy-plane is defined as

$$\iint_R f(x,\, y)dA = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*,\, y_{ij}^*)\Delta A. \tag{5.1}$$

$dA = dx\,dy = dy\,dx$

$dV = dx\,dy\,dz$

$$\int_1^2 \int_0^1 xy\, dx\, dy$$

y constant

$$\int_1^2 \left. \frac{x^2}{2} y \right|_0^1 dy = \int_1^2 \frac{y}{2}\, dy = \left. \frac{y^2}{4} \right|_1^2 = \frac{3}{4}$$

Theorem 5.1: Properties of Double Integrals

Assume that the functions $f(x, y)$ and $g(x, y)$ are integrable over the rectangular region R; S and T are subregions of R; and assume that m and M are real numbers.

i. The sum $f(x, y) + g(x, y)$ is integrable and

$$\iint_R [f(x, y) + g(x, y)]dA = \iint_R f(x, y)dA + \iint_R g(x, y)dA.$$

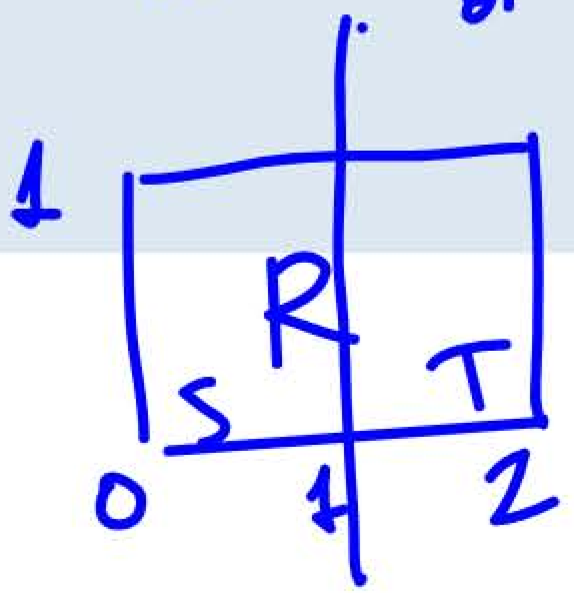
ii. If c is a constant, then $cf(x, y)$ is integrable and

$$\iint_R cf(x, y)dA = c\iint_R f(x, y)dA.$$

iii. If $R = S \cup T$ and $S \cap T = \varnothing$ except an overlap on the boundaries, then

$$\iint_R f(x, y)dA = \iint_S f(x, y)dA + \iint_T f(x, y)dA.$$

$$\int_a^b f(x)dx = \int_a^c f\,dx + \int_c^b f\,dx.$$



iv. If $f(x, y) \geq g(x, y)$ for (x, y) in R, then

$$\iint_R f(x, y)dA \geq \iint_R g(x, y)dA.$$

area of R

v. If $m \leq f(x, y) \leq M$, then

min Max

$$m \times A(R) \leq \iint\limits_R f(x, y)\,dA \leq M \times A(R).$$

vi. In the case where $f(x, y)$ can be factored as a product of a function $g(x)$ of x only and a function $h(y)$ of y only, then over the region $R = \{(x, y) | a \leq x \leq b,\ c \leq y \leq d\}$, the double integral can be written as

$$\iint\limits_R f(x, y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

Definition

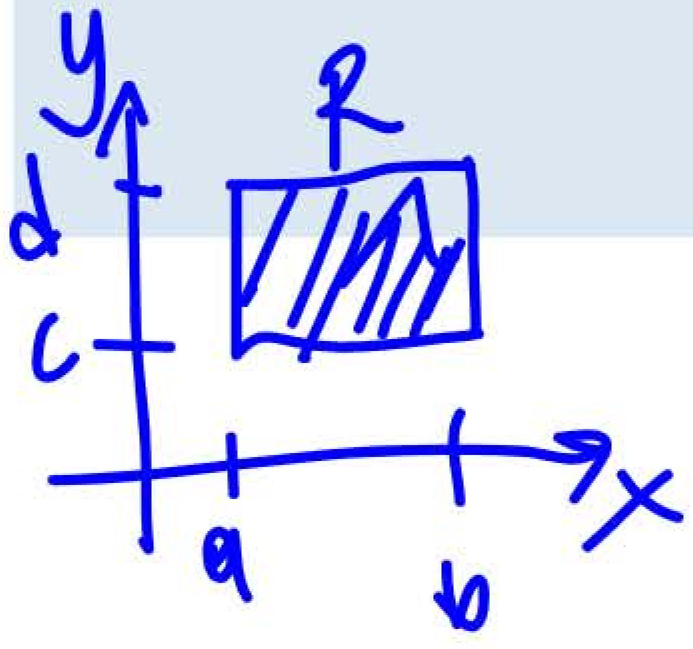
Assume $a, b, c,$ and d are real numbers. We define an **iterated integral** for a function $f(x, y)$ over the rectangular region $R = [a, b] \times [c, d]$ as

a.

$$\int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx \tag{5.2}$$

b.

$$\int_c^d \int_a^b f(x, y)\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx \right] dy. \tag{5.3}$$



$$\int_2^3 \int_0^1 xy\,dx\,dy = \int_0^1 \int_2^3 xy\,dy\,dx$$

Theorem 5.2: Fubini's Theorem

Suppose that $f(x, y)$ is a function of two variables that is continuous over a rectangular region $R = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b, \ c \le y \le d\}$. Then we see from **Figure 5.7** that the double integral of f over the region equals an iterated integral,

$$\iint_R f(x, y)\,dA = \iint_R f(x, y)\,dx\,dy = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy.$$

More generally, **Fubini's theorem** is true if f is bounded on R and f is discontinuous only on a finite number of continuous curves. In other words, f has to be integrable over R.

We can change the order of integration

dxdy dydx

Slide 8

5.2 a. Use the properties of the double integral and Fubini's theorem to evaluate the integral

$$\int_0^1 \int_{-1}^3 (3 - x + 4y)\,dy\,dx.$$

b. Show that $0 \leq \iint_R \sin \pi x \cos \pi y \, dA \leq \frac{1}{32}$ where $R = \left(0, \frac{1}{4}\right)\left(\frac{1}{4}, \frac{1}{2}\right)$. $\rightarrow A(R) = \frac{1}{4} \times \frac{1}{4}$

a) $\int_0^1 \left[(3-x)y + 2y^2\right]\Big|_{-1}^{3} dx = \int_0^1 \left((3-x)4 + 16\right) dx = \int_0^1 (28 - 4x)\,dx$

$= \left(28x - 2x^2\right)\Big|_0^1 = 28 - 2 = 26.$

$\int_{-1}^{3}\int_0^1 (3 - x + 4y)\,dx\,dy = \int_{-1}^{3} \left(3x - \frac{x^2}{2} + 4yx\right)\Big|_0^1 dy = \int_{-1}^{3} \left(\frac{5}{2} + 4y\right) dy$

$= \left(\frac{5}{2}y + 2y^2\right)\Big|_{-1}^{3} = \frac{15}{2} + 18 - \left(-\frac{5}{2} + 2\right)$

$= 10 + 18 - 2 = \underline{26}$

5.3 Evaluate $\int_{y=-3}^{y=2}\int_{x=3}^{x=5}\left(2-3x^2+y^2\right)dx\,dy.$

$$\int_{-3}^{2}\left(2x-x^3+y^2x\right)\Big|_3^5\,dy=\int_{-3}^{2}\left[10-125+5y^2-\left(6-27+3y^2\right)\right]dy$$

$$=\int_{-3}^{2}\left(-94+2y^2\right)dy=\left(-94y+\frac{2y^3}{3}\right)\Big|_{-3}^{2}$$

$$=\left(-188+\frac{16}{3}\right)-\left(282-\frac{54}{3}\right)=\frac{70}{3}-470$$

$$=\frac{70-1410}{3}=-\frac{1340}{3}$$

5.4 Evaluate the integral $\iint_R xe^{xy}\,dA$ where $R = [0, 1] \times [0, \ln 5]$.

x y

$$\int e^{5x}dx = \frac{e^{5x}}{5}$$

$$\int e^{xy}dy = \frac{e^{xy}}{x}$$

$$\int_0^1\int_0^{\ln 5} xe^{xy}\,dy\,dx = \int_0^1 \frac{\cancel{x}e^{xy}}{\cancel{x}}\Big|_0^{\ln 5} dx$$

$$= \int_0^1 \left[e^{\ln 5 \cdot x} - e^{x\cdot 0}\right]dx = \int_0^1 (5e^x - 1)\,dx = (5e^x - x)\Big|_0^1 = 5e - 6$$

$$\int_0^{\ln 5}\int_0^1 xe^{xy}\,dx\,dy$$

$$\int_0^1 xe^x\,dx$$

Integration by parts...

$$\int x\ln x\,dx \qquad \int x\sin x\,dx$$

Slide 11

Definition

The area of the region R is given by $A(R) = \iint_R 1\,dA.$

$f(x,y) = 1 = \text{height}$ Volume = Area

In the following exercises, evaluate the iterated integrals by choosing the order of integration.

$(\ln(x^2+1))' = \dfrac{2x}{x^2+1}$

33. $\displaystyle\int_0^1 \int_1^2 \left(\frac{x}{x^2+y^2}\right) dy\,dx = \int_1^2 \int_0^1 \frac{x}{x^2+y^2}\,dx\,dy$

$\displaystyle\int_1^2 \frac{\ln(x^2+y^2)}{2}\Big|_0^1\,dy = \int_1^2 \frac{\ln(1+y^2) - \ln(y^2)}{2}\,dy$

5.5 Find the volume of the solid bounded above by the graph of $f(x, y) = xy\sin(x^2 y)$ and below by the xy-plane on the rectangular region $R = [0, 1] \times [0, \pi]$.

positive on R

$$V = \iint_R f(x,y)\,dA = \int_0^{\pi}\int_0^1 xy\sin(x^2y)\,dx\,dy$$

$$\frac{\partial(\cos(x^2y))}{\partial x} = -\sin(x^2y)\,2xy$$

$$\int_0^{\pi} -\frac{\cos(x^2y)}{2}\Big|_0^1 dy$$

$$= \int_0^{\pi} \frac{-\cos y + 1}{2}\,dy = \left(\frac{-\sin y + y}{2}\right)\Big|_0^{\pi} = \frac{\pi}{2}$$

volume is $\frac{\pi}{2}$ unit^3.

If we prefer $\int_0^1\int_0^{\pi} xy\sin(x^2y)\,dy\,dx$ Integration by parts -

Slide 13

Definition

The average value of a function of two variables over a region R is

$$f_{\text{ave}} = \frac{1}{\text{Area } R} \iint_R f(x, y)\,dA. \quad (5.4)$$

$[a,b]$

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

In the following exercises, find the average value of the function over the given rectangles.

36. $f(x, y) = x^4 + 2y^3, \quad R = [1, 2] \times [2, 3]$

$A(R) = 1$

$\int_1^2\int_2^3 (x^4+2y^3)\,dy\,dx = \int_1^2 \left(x^4 y + \frac{y^4}{2}\right)\Big|_2^3 dx$

$= \int_1^2 \left(x^4 + \frac{81}{2} - \frac{16}{2}\right) dx = \left(\frac{x^5}{5} + \frac{65x}{2}\right)\Big|_1^2 = \left(\frac{32}{5} + 65\right) - \left(\frac{1}{5} + \frac{65}{2}\right)$

$= \frac{31}{5} + \frac{65}{2} = \frac{62+325}{10}$

$f_{ave} = 387/10$

Slide 14

5.2 | Double Integrals over General Regions

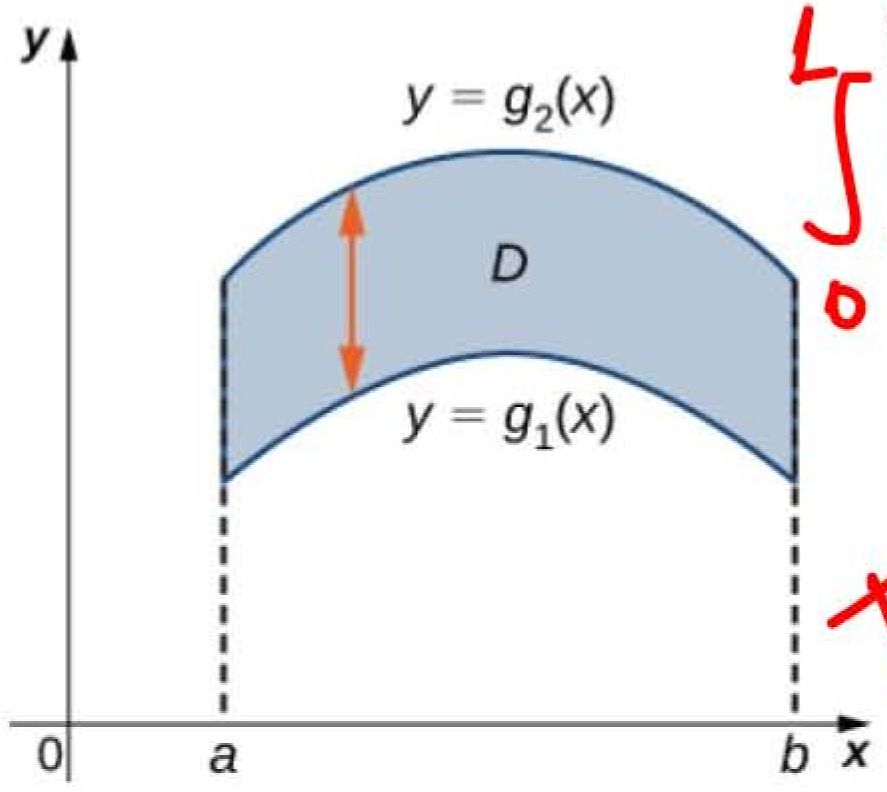
Definition

A region D in the (x, y)-plane is of **Type I** if it lies between two vertical lines and the graphs of two continuous functions $g_1(x)$ and $g_2(x)$. That is (Figure 5.13),

$$D = \{(x, y) | a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

A region D in the xy plane is of **Type II** if it lies between two horizontal lines and the graphs of two continuous functions $h_1(y)$ and $h_2(y)$. That is (Figure 5.14),

$$D = \{(x, y) | \underline{c \le y \le d},\ h_1(y) \le x \le h_2(y)\}.$$

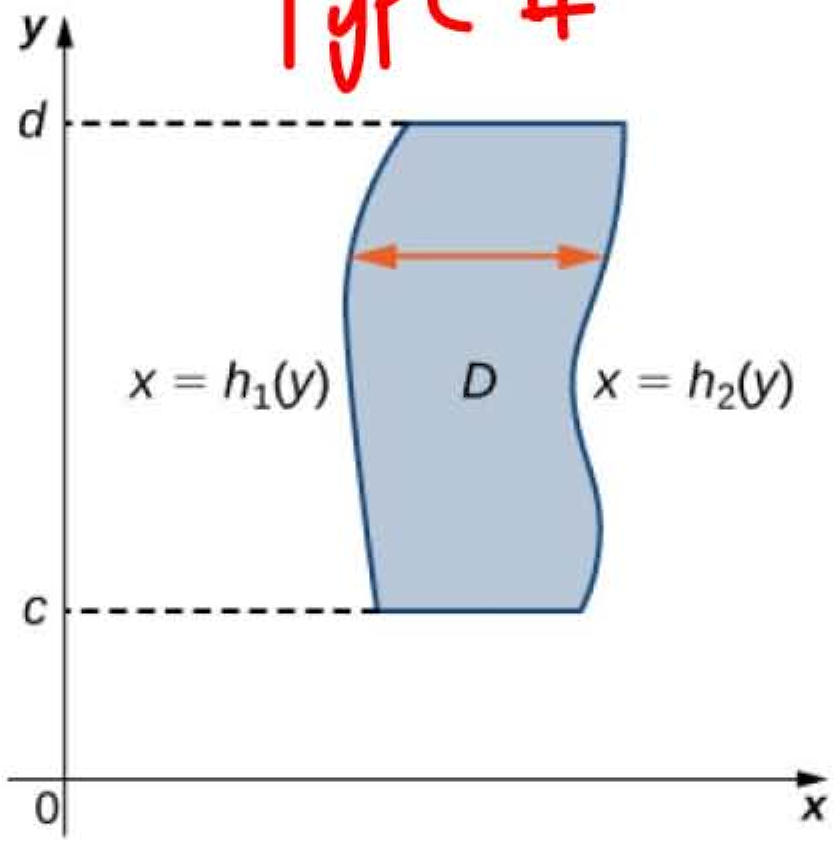


$$\int_0^1 \int_x^{2x} - \,dy\,dx$$

$$\int_2^4 \int_y^{y^2} - \,dx\,dy$$

Type I

Type II

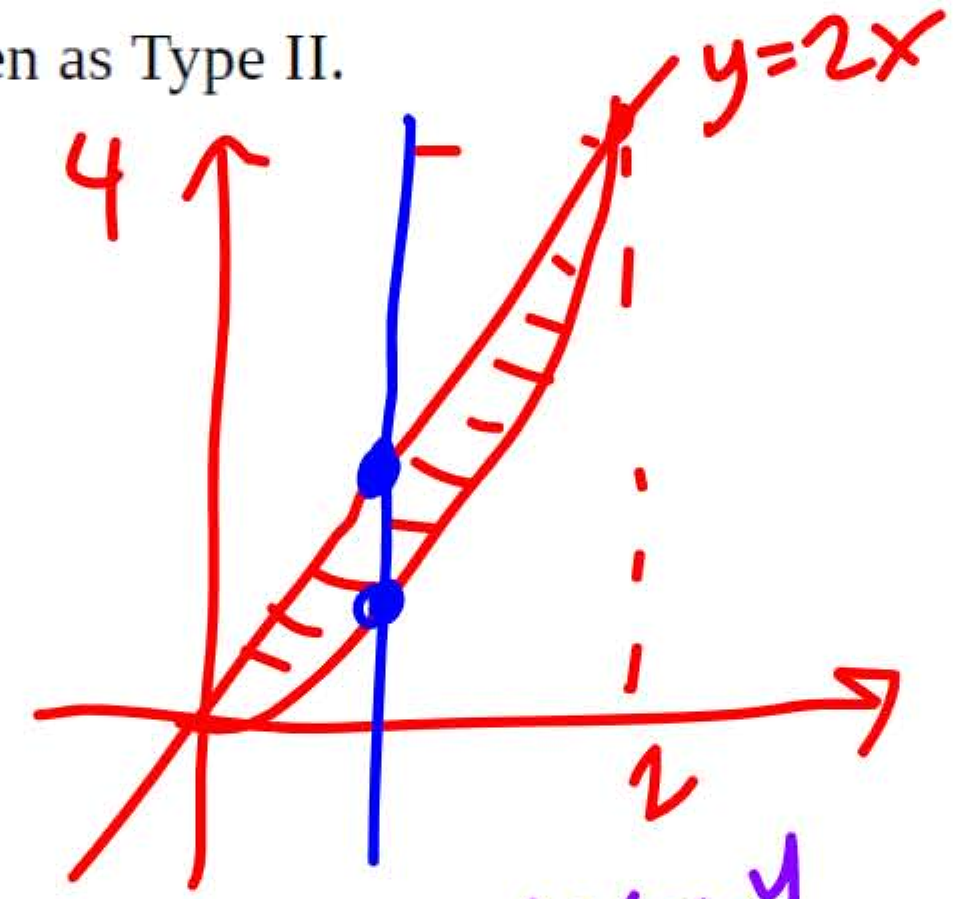


5.7 Consider the region in the first quadrant between the functions $y = 2x$ and $y = x^2$. Describe the region first as Type I and then as Type II.

$2x = x^2$

$0 = x^2 - 2x$

$= x(x-2)$

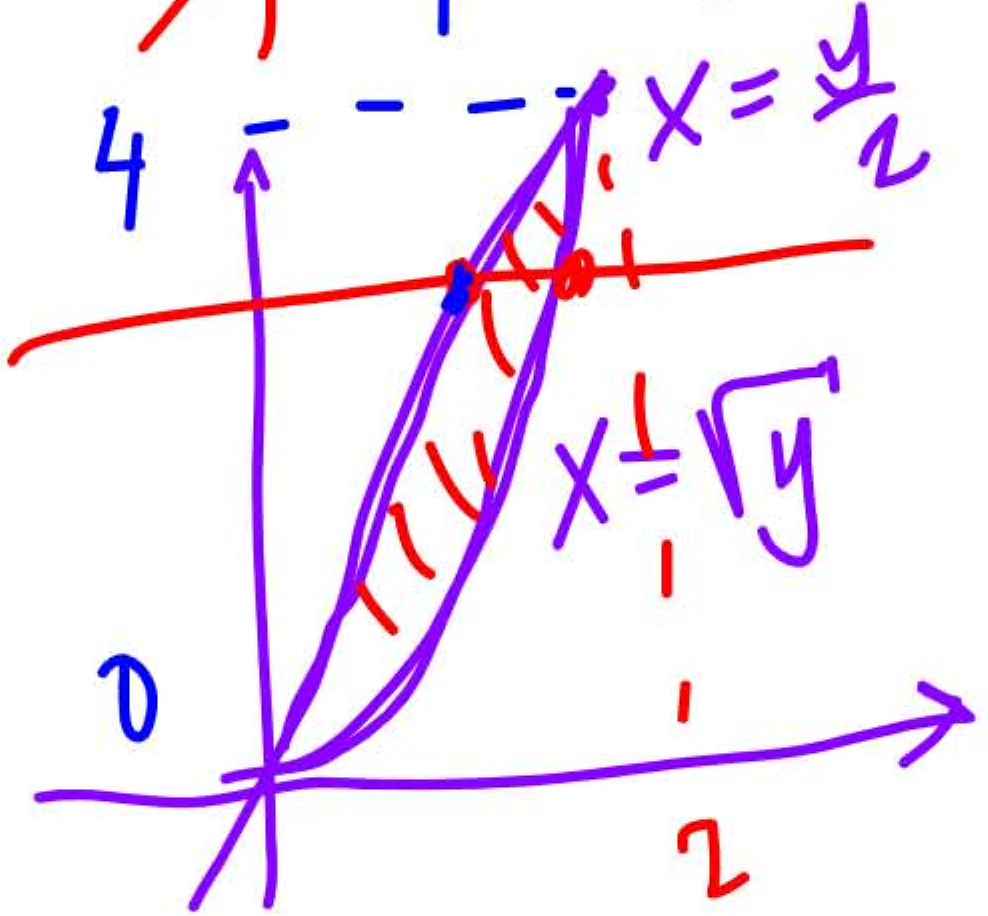


$0 \le x \le 2, \quad x^2 \le y \le 2x$

$$\int_0^2 \int_{x^2}^{2x} f(x,y)\, dy\, dx$$

Type I

Type II



$0 \le y \le 4, \quad \frac{y}{2} \le x \le \sqrt{y}$

$$\int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y)\, dx\, dy$$

Theorem 5.3: Double Integrals over Nonrectangular Regions

Suppose $g(x, y)$ is the extension to the rectangle R of the function $f(x, y)$ defined on the regions D and R as shown in **Figure 5.12** inside R. Then $g(x, y)$ is integrable and we define the double integral of $f(x, y)$ over D by

$$\iint_D f(x, y)dA = \iint_R g(x, y)dA.$$

Theorem 5.4: Fubini's Theorem (Strong Form)

For a function $f(x, y)$ that is continuous on a region D of Type I, we have

we may change

$$\iint_D f(x, y)dA = \iint_D f(x, y)dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)dy\right]dx. \quad (5.5)$$

Type I
the order
of integration

Similarly, for a function $f(x, y)$ that is continuous on a region D of Type II, we have

$$\iint_D f(x, y)dA = \iint_D f(x, y)dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)dx\right]dy. \quad (5.6)$$

Evaluating an Iterated Integral over a Type II Region

Evaluate the integral $\iint\limits_D \left(3x^2 + y^2\right)dA$ where $D = \left\{(x, y) \mid -2 \le y \le 3,\ y^2 - 3 \le x \le y + 3\right\}$.

Type II

$$\int_{-2}^{3}\int_{y^2-3}^{y+3}\left(3x^2+y^2\right)dx\,dy = \int_{-2}^{3}\left(x^3+y^2x\right)\Big|_{y^2-3}^{y+3}dy$$

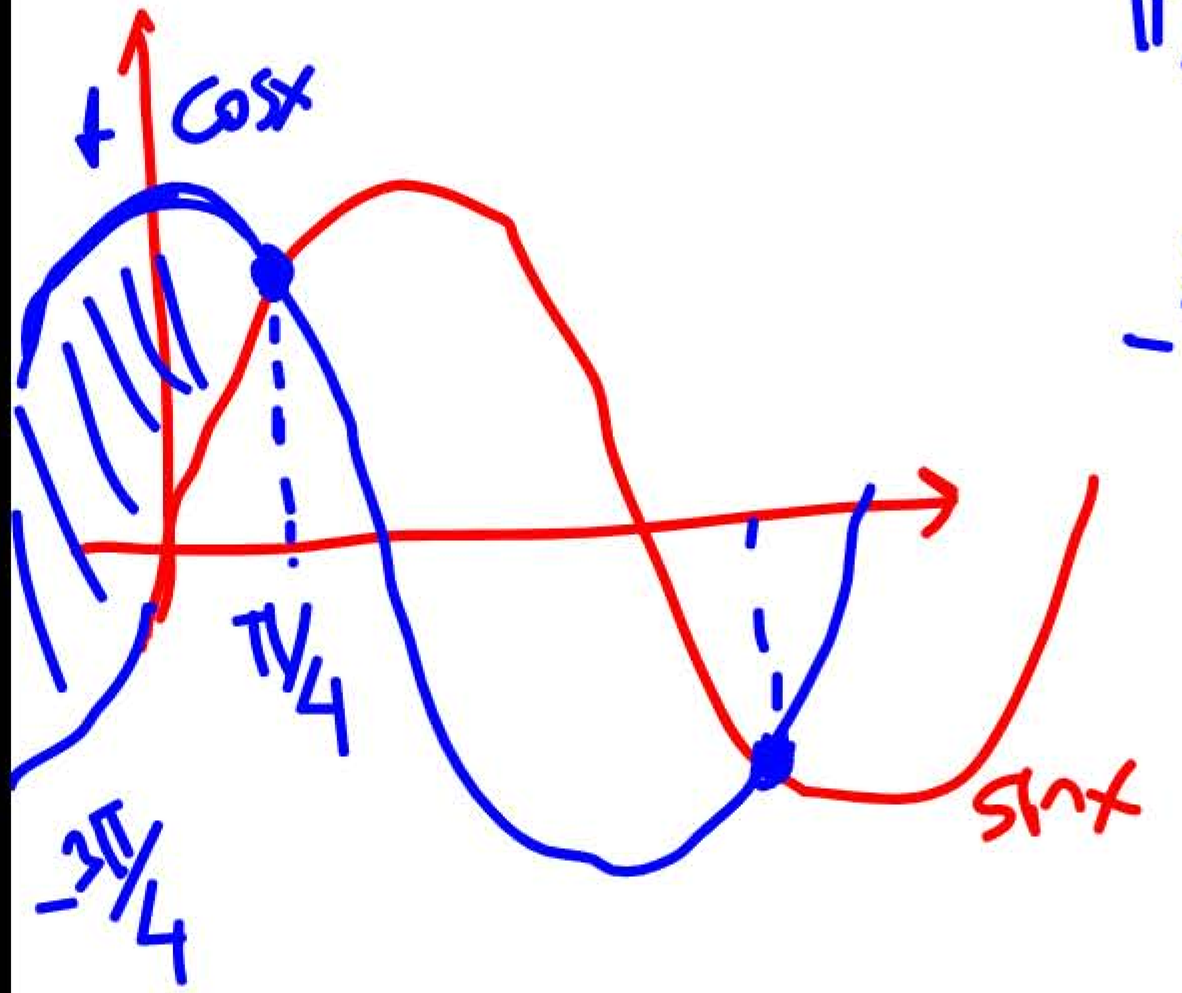
y x

$$=\int_{-2}^{3}\left((y+3)^3+y^2(y+3)-(y^2-3)^3+y^2(y^2-3)\right)dy$$

$$=\int_{-2}^{3}\left(y^3+9y^2+27y+\cancel{27}+y^3+\cancel{3y^2}-y^6+9y^4-27y^2-\cancel{27}+y^4-\cancel{3y^2}\right)dy$$

$$=\left(-\frac{y^7}{7}+\frac{2y^4}{4}-6y^3+2y^5+\frac{27y^2}{2}\right)\Big|_{-2}^{3}$$

5.8 Sketch the region D and evaluate the iterated integral $\iint_D xy\,dy\,dx$ where D is the region bounded by the curves $y = \cos x$ and $y = \sin x$ in the interval $[-3\pi/4, \pi/4]$.



$$\int_{-3\pi/4}^{\pi/4}\int_{\sin x}^{\cos x} xy\,dy\,dx = \int_{-3\pi/4}^{\pi/4} x\frac{y^2}{2}\Big|_{\sin x}^{\cos x}\,dx$$

$$\int_{-3\pi/4}^{\pi/4} \frac{x}{2}(\cos^2 x - \sin^2 x)\,dx = \int_{-3\pi/4}^{\pi/4} \frac{x}{2}\cos 2x\,dx$$

integration by parts.

or formula -

$D:\ \sin x \le y \le \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

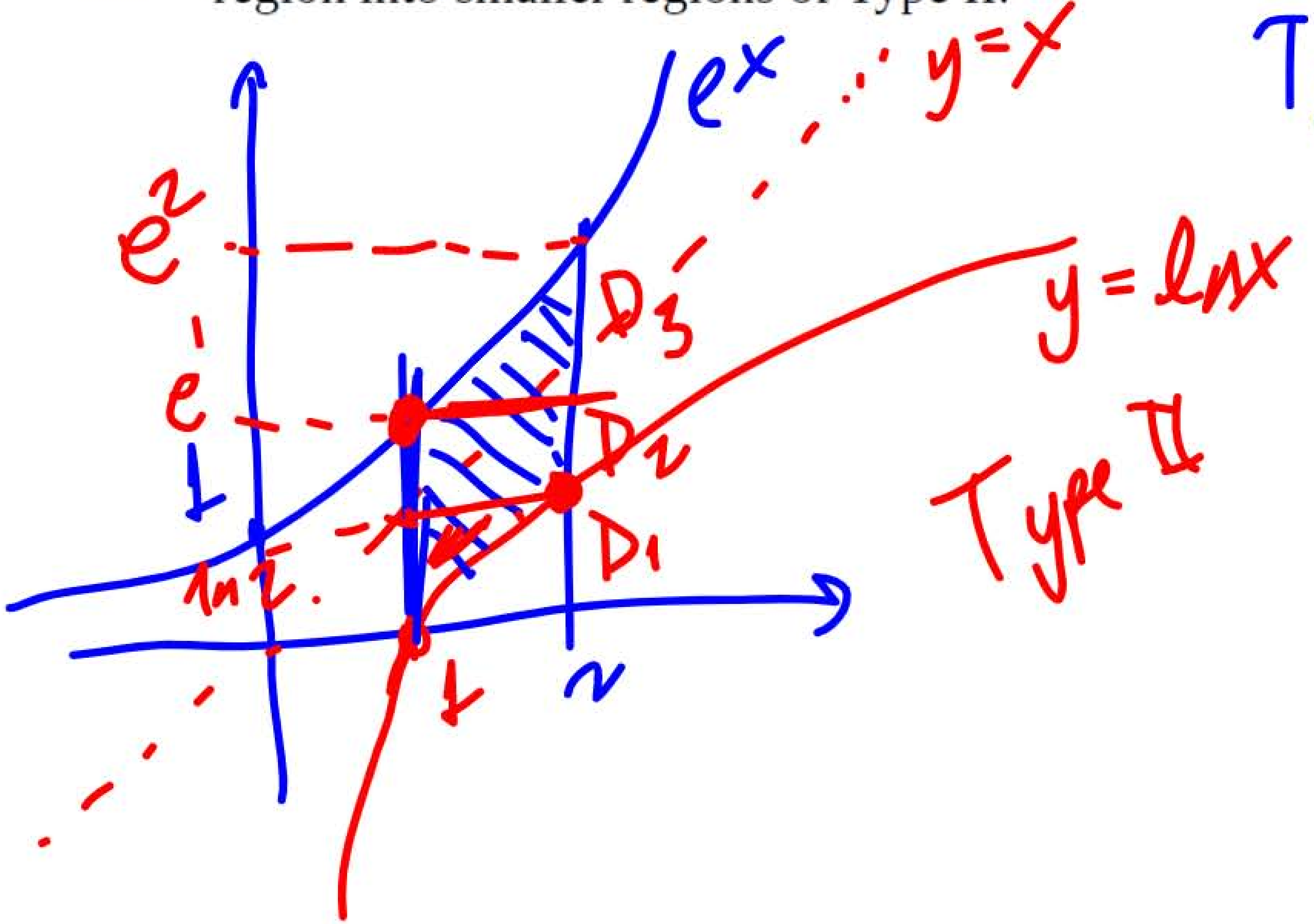
Theorem 5.5: Decomposing Regions into Smaller Regions

Suppose the region D can be expressed as $D = D_1 \cup D_2$ where D_1 and D_2 do not overlap except at their boundaries. Then

$$\iint\limits_D f(x, y)dA = \iint\limits_{D_1} f(x, y)dA + \iint\limits_{D_2} f(x, y)dA. \tag{5.7}$$

5.9 Consider the region bounded by the curves $y = \ln x$ and $y = e^x$ in the interval $[1, 2]$. Decompose the region into smaller regions of Type II.

Type I, $1 \leqslant x \leqslant 2$, $\ln x \leqslant y \leqslant e^x$



Type II

D_1 $0 \leqslant y \leqslant \ln 2$, $1 \leqslant x \leqslant e^y$

D_2 $\ln 2 \leqslant y \leqslant e$, $1 \leqslant x \leqslant 2$

D_3 $e \leqslant y \leqslant e^2$, $\ln y \leqslant x \leqslant 2$